Tail behavior of solutions of linear recursions on trees

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Abstract

Consider the linear nonhomogeneous fixed-point equation

$$R \stackrel{\mathcal{D}}{=} \sum_{i=1}^{N} C_i R_i + Q,$$

where $(Q, N, C_1, C_2, \ldots)$ is a random vector with $N \in \{0, 1, 2, 3, \ldots\} \cup \{\infty\}$, $C_i \geq 0$ for all $i \in \mathbb{N}$, $P(|Q| > 0) > 0$, and $\{R_i\}_{i \in \mathbb{N}}$ is a sequence of i.i.d. random variables independent of $(Q, N, C_1, C_2, \ldots)$ having the same distribution as R. It is known that R will have a heavy-tailed distribution under several different sets of assumptions on the vector $(Q, N, C_1, C_2, \ldots)$. This paper investigates the settings where either $Z_N = \sum_{i=1}^{N} C_i$ or Q are regularly varying with index $-\alpha < -1$ and $E\left[\sum_{i=1}^{N} C_i^{\alpha}\right] < 1$. This work complements previous results showing that $P(R > t) \sim H t^{-\alpha}$ provided there exists a solution $\alpha > 0$ to the equation $E\left[\sum_{i=1}^{N} |C_i|^{\alpha}\right] = 1$, and both Q and Z_N have lighter tails.

1. Introduction

Motivated by the analysis of information ranking algorithms, this paper investigates the tail behavior of the solution to the stochastic fixed-point equation

$$R \stackrel{\mathcal{D}}{=} \sum_{i=1}^{N} C_i R_i + Q, \tag{1}$$

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where $(Q, N, C_1, C_2, \ldots)$ is a random vector with $N \in \mathbb{N} \cup \{\infty\}$, $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, $C_i \geq 0$ for all $i \in \mathbb{N}$, $P(|Q| > 0) > 0$, and $\{R_i\}_{i \in \mathbb{N}}$ is a sequence of i.i.d. random variables independent of $(Q, N, C_1, C_2, \ldots)$ having the same distribution as R; the symbol $\overset{\mathcal{D}}{=}$ denotes equality in distribution. This stochastic fixed-point equation recently appeared in the analysis of Google's PageRank algorithm [8], which computes the ranks of pages on the World Wide Web according to the recursion

$$\mathrm{PR}(p_i) = \frac{1 - d}{n} + d \sum_{p_j \in M(p_i)} \frac{\mathrm{PR}(p_j)}{L(p_j)}, \tag{2}$$

where $p_1, p_2, \ldots, p_n$ are the pages under consideration, $M(p_i)$ is the set of pages that link to p_i, $L(p_j)$ is the number of outbound links on page p_j, $\mathrm{PR}(p_j)$ is the PageRank of page p_j, $d \in (0, 1)$ is a constant known as the "damping factor", and n is the total number of pages. A first order stochastic approximation for the rank of a randomly chosen page is obtained by multiplying both sides of (2) by n and considering the fixed-point equation

$$R \overset{\mathcal{D}}{=} 1 - d + d \sum_{i=1}^{N} \frac{R_j}{D_j},$$

where N is a random variable distributed according to the in-degree distribution of the web graph, $\{D_j\}$ are i.i.d. random variables distributed according to the *effective out-degree* distribution (see [27, Section 3.2] for more details), and $\{R_j\}$ are i.i.d. random variables having the same distribution as R. This approach, first introduced in [27], can be thought of as approximating the web graph with a branching tree, a well known technique used in the analysis of random graphs (see, e.g., [25] and the references therein).

The fixed-point equation (1) has been recently analyzed in [14,26] for the special case of $Q, N, \{C_i\}$ nonnegative and mutually independent, with the $\{C_i\}$ i.i.d.; in [26] the pair (Q, N) was allowed to be dependent under stronger moment conditions. One of the results in these articles was that when the distribution of N is heavy-tailed, in particular, regularly varying, and Q has lighter tails than N, the tail distribution of R is proportional to that of N, i.e.,

$$P(R > x) \sim H P(N > x) \quad \text{as } x \to \infty,$$

where $f(x) \sim g(x)$ is used throughout the paper to denote $\lim_{x \to \infty} f(x)/g(x) = 1$. This indicates that one way in which a page can achieve a very high rank is by having a very large in-degree; this is indeed consistent with the large-deviations analysis of the weighted random sum $\sum_{i=1}^{N} C_i R_i$ (see [22] for more details). Since the in-degree of a page is something that individuals can easily modify (e.g. by creating many fake links), one can argue that there is a need to develop more robust ranking algorithms. One way in which this can be done is by choosing a different set of weights $\{C_i\}$ in (1), e.g., different from those of PageRank, $C_j = d/D_j$. The general setting of $(Q, N, C_1, C_2, \ldots)$ arbitrarily dependent with the $\{C_i\}$ not necessarily independent and/or identically distributed allows a great level of flexibility in this respect. This setting is also consistent with the broader literature on weighted branching processes [23] and branching random walks [4], which appear in the probabilistic analysis of other algorithms as well [24,21], e.g. Quicksort algorithm [9].

A very well known special case of Eq. (1) is obtained by setting $N \equiv 1$, since then it becomes the stochastic recurrence equation

$$R \overset{\mathcal{D}}{=} CR + Q, \quad (C, Q) \text{ independent of } R.$$

The power law tail asymptotics of the solution R to this equation were established in the classical work of Kesten [18] (in a multivariate setting), and were also derived through the use of implicit renewal theory by Goldie [10]. The approach from [10] was generalized in [15] to analyze (1) for real-valued weights $\{C_i\}$. The main assumption in [15] (and the corresponding $N \equiv 1$ versions of [18,10]) is the existence of a solution $\alpha > 0$ to the equation $E\left[\sum_{i=1}^{N} |C_i|^{\alpha}\right] = 1$ such that $E\left[\sum_{i=1}^{N} |C_i|^{\alpha} \log |C_i|\right] > 0, E[|Q|^{\alpha}] < \infty$, and if $\alpha > 1, E\left[\sum_{i=1}^{N} |C_i|\right] < 1, E\left[\left(\sum_{i=1}^{N} |C_i|\right)^{\alpha}\right] < \infty$, in which case

$$P(R > x) \sim Hx^{-\alpha} \quad \text{as } x \to \infty,$$

for some constant $H \geq 0$. The work of [14] already shows that if such α does not exist, then $P(R > x)$ can still be regularly varying if either the distribution of N or Q are regularly varying. When $N \equiv 1, (C, Q)$ are generally dependent, $C \geq 0$ a.s. and Q is regularly varying, the tail equivalence of $P(R > x)$ and $P(Q > x)$ was shown in [12]. The main results in this paper, Theorems 3.4 and 4.4, give the corresponding generalization of the results in [12,14] to arbitrarily dependent $(Q, N, C_1, C_2, \ldots)$. In particular, it is shown that if either $P\left(\sum_{i=1}^{N} C_i > x\right)$, or $P(Q > x)$, are regularly varying with index $-\alpha < -1$, and certain moment conditions are satisfied, then

$$P(R > x) \sim H' P\left(\sum_{i=1}^{N} C_i > x\right), \quad \text{respectively,} \quad P(R > x) \sim H'' P(Q > x)$$

as $x \to \infty$, for some explicit constants $H', H'' > 0$. We point out that (1) may also have light-tailed solutions, as the work in [11] shows for the $N \equiv 1$ case, but we focus here only on the heavy-tailed ones.

The paper is organized as follows. First we construct an explicit solution to (1) on a weighted branching tree. As will be discussed in more detail in Section 2, this particular solution is the only one of practical interest, since under mild technical conditions, this is the unique limit of the process that results from the iteration of (1) (see Lemma 2.4). The main result for the case where the tail behavior of R is dominated by the sum of the weights, $\sum_{i=1}^{N} C_i$, is given in Section 3, and the main result for the case where Q dominates is given in Section 4. The main technical contribution of the paper is in the derivation of uniform bounds (in n and x) for the distribution of the sum of the weights in the nth generation of a weighted branching tree, $P(W_n > x)$, given in Propositions 3.2 and 4.2. These uniform bounds are the key tool in establishing the geometric rate of convergence of the iterations of the fixed-point equation (1) to the solution R constructed in Section 2. Finally, the more technical proofs are postponed to Section 5 and the Appendix.

The last thing to mention is that the approach used to derive the uniform bounds from Propositions 3.2 and 4.2 can also be helpful in the analysis of other recursions on trees, such as the ones studied in [16] and the more extensive survey of [1], e.g.,

$$R \overset{\mathcal{D}}{=} \left(\bigvee_{i=1}^{N} C_i R_i\right) \vee Q \qquad R \overset{\mathcal{D}}{=} \left(\bigvee_{i=1}^{N} C_i R_i\right) + Q,$$

that may fall outside of the implicit renewal theory framework of [16].

2. Construction of a solution on a tree

We start by constructing in this section a particular solution to the fixed-point equation

$$R \stackrel{\mathcal{D}}{=} \sum_{i=1}^{N} C_i R_i + Q, \tag{3}$$

where $(Q, N, C_1, C_2, \ldots)$ is a random vector with $N \in \mathbb{N} \cup \{\infty\}$, $C_i \geq 0$ for all $i \in \mathbb{N}$, $P(|Q| > 0) > 0$, and $\{R_i\}_{i \in \mathbb{N}}$ is a sequence of i.i.d. real-valued random variables independent of $(Q, N, C_1, C_2, \ldots)$ having the same distribution as R. We will show in Section 2.2 that the process that results from iterating (3) converges under mild conditions to this particular solution.

First we construct a random tree $\mathcal{T}$. Let $\mathbb{N}_+ = \{1, 2, 3, \ldots\}$ be the set of positive integers and let $U = \bigcup_{k=0}^{\infty} (\mathbb{N}_+)^k$ be the set of all finite sequences of positive integers $\mathbf{i} = (i_1, i_2, \ldots, i_n)$, where by convention $\mathbb{N}_+^0 = \{\emptyset\}$ contains the null sequence $\emptyset$. We use the notation $\emptyset$ to denote the root node of $\mathcal{T}$, and $A_n, n \geq 0$, to denote the set of all individuals in the nth generation of $\mathcal{T}$, $A_0 = \{\emptyset\}$. Also, for $\mathbf{i} \in A_1$ we simply use the notation $\mathbf{i} = i_1$, that is, without the parenthesis. Similarly, for $\mathbf{i} = (i_1, \ldots, i_n)$ we will use $(\mathbf{i}, j) = (i_1, \ldots, i_n, j)$ to denote the index concatenation operation, if $\mathbf{i} = \emptyset$, then $(\mathbf{i}, j) = j$.

Next, let Z_n be the number of individuals in the nth generation of $\mathcal{T}$, that is, $Z_n = |A_n|$, where $|\cdot|$ denotes the cardinality of a set; in particular, $Z_0 = 1$. We iteratively construct the tree as follows. Let N be the number of individuals born to the root node $\emptyset$, $N_\emptyset = N$, and let $\{N_\mathbf{i}\}_{\mathbf{i} \in U}$ be i.i.d. copies of N. Define now

$$A_1 = \{i \in \mathbb{N}_+ : i \leq N\}, \qquad A_n = \{(\mathbf{i}, i_n) \in U : \mathbf{i} \in A_{n-1}, i_n \leq N_\mathbf{i}\}. \tag{4}$$

It follows that the number of individuals $Z_n = |A_n|$ in the nth generation, $n \geq 1$, satisfies the branching recursion

$$Z_n = \sum_{\mathbf{i} \in A_{n-1}} N_\mathbf{i}.$$

Now, we construct the weighted branching tree $\mathcal{T}_{Q,C}$ as follows. The root node $\emptyset$ is assigned a vector $(Q_\emptyset, N_\emptyset, C_{(\emptyset,1)}, C_{(\emptyset,2)}, \ldots) = (Q, N, C_1, C_2, \ldots)$ with $N \in \mathbb{N} \cup \{\infty\}$ and $P(|Q| > 0) > 0$; N determines the number of nodes in the first generation of $\mathcal{T}$ according to (4). Each node $\mathbf{i} \in A_n, n \geq 1$, is then assigned an i.i.d. copy $(Q_\mathbf{i}, N_\mathbf{i}, C_{(\mathbf{i},1)}, C_{(\mathbf{i},2)}, \ldots)$ of the root vector which is used to construct $A_{n+1} = \{(\mathbf{i}, i_{n+1}) \in U : \mathbf{i} \in A_n, i_{n+1} \leq N_\mathbf{i}\}$. Note that the vectors $(Q_\mathbf{i}, N_\mathbf{i}, C_{(\mathbf{i},1)}, C_{(\mathbf{i},2)}, \ldots)$, $\mathbf{i} \in A_n$, are chosen independently of all the previously assigned vectors $(Q_\mathbf{j}, N_\mathbf{j}, C_{(\mathbf{j},1)}, C_{(\mathbf{j},2)}, \ldots)$, $\mathbf{j} \in A_k$, $0 \leq k \leq n - 1$. For each node in $\mathcal{T}_{Q,C}$ we also define the weight $\Pi_{(i_1,\ldots,i_n)}$ via the recursion

$$\Pi_{i_1} = C_{i_1}, \qquad \Pi_{(i_1,\ldots,i_n)} = C_{(i_1,\ldots,i_n)} \Pi_{(i_1,\ldots,i_{n-1})}, \quad n \geq 2,$$

where $\Pi = 1$ is the weight of the root node. Note that the weight $\Pi_{(i_1,\ldots,i_n)}$ is equal to the product of all the weights $C_{(\cdot)}$ along the branch leading to node $(i_1, \ldots, i_n)$, as depicted in Fig. 1.

We now formally define on the weighted branching tree $\mathcal{T}_{Q,C}$ the process

$$W_0 = Q, \qquad W_n = \sum_{\mathbf{i} \in A_n} Q_\mathbf{i} \Pi_\mathbf{i}, \quad n \geq 1, \tag{5}$$
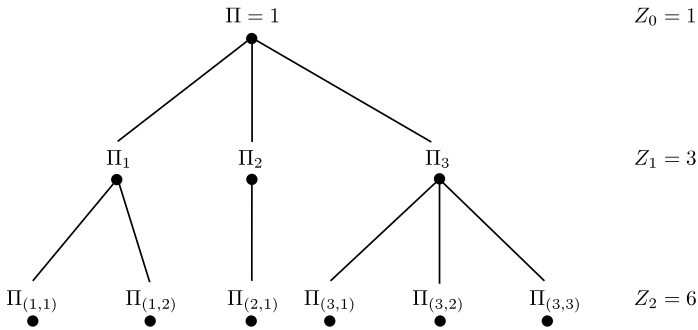
Fig. 1. Weighted branching tree.

and the process $\{R^{(n)}\}_{n \geq 0}$ according to

$$R^{(n)} = \sum_{k=0}^{n} W_k, \quad n \geq 0. \tag{6}$$

Note that $R^{(n)}$ is the sum of the weights of all the nodes on the tree up to the nth generation. In general, since the $\{Q_i\}$ are allowed to be real-valued, neither W_n nor $R^{(n)}$ are necessarily well-defined; throughout the rest of the paper we will always impose sufficient conditions on $(Q, N, C_1, C_2, \ldots)$ to ensure the absolute convergence of W_n, but before we do that (5) and (6) are to be interpreted formally.

It is not hard to see that $R^{(n)}$ satisfies the recursion

$$R^{(n)} = \sum_{j=1}^{N_\emptyset} C_{(\emptyset,j)} R_j^{(n-1)} + Q_\emptyset = \sum_{j=1}^{N} C_j R_j^{(n-1)} + Q, \quad n \geq 1, \tag{7}$$

where $\{R_j^{(n-1)}\}$ are independent copies of $R^{(n-1)}$ corresponding to the tree starting with individual j in the first generation and ending on the nth generation; note that $R_j^{(0)} = Q_j$. Moreover, since the tree structure repeats itself after the first generation, W_n satisfies

$$
\begin{aligned}
W_n &= \sum_{\mathbf{i} \in A_n} Q_{\mathbf{i}} \Pi_{\mathbf{i}} \\
&= \sum_{k=1}^{N_\emptyset} C_{(\emptyset,k)} \sum_{(k,\ldots,i_n) \in A_n} Q_{(k,\ldots,i_n)} \prod_{j=2}^{n} C_{(k,\ldots,i_j)} \\
&\stackrel{\mathcal{D}}{=} \sum_{k=1}^{N} C_k W_{(n-1),k},
\end{aligned}
\tag{8}
$$

where $\{W_{(n-1),k}\}$ is a sequence of i.i.d. random variables independent of $(N, C_1, C_2, \ldots)$ and having the same distribution as W_{n-1}.

The following result from [15, Lemma 4.1] gives the convergence of $R^{(n)}$ to a proper limit; $\stackrel{\triangle}{=}$ denotes definition.

Lemma 2.1. *If for some $0 < \beta \le 1$, $E[|Q|^\beta] < \infty$ and $E\left[\sum_{j=1}^N C_j^\beta\right] < 1$, then $R^{(n)} \to R$ a.s. as $n \to \infty$, where $E[|R|^\beta] < \infty$ and is given by*

$$R \triangleq \sum_{n=0}^{\infty} W_n. \tag{9}$$

As discussed in [15], the observation that under the conditions of the lemma the sum of all the absolute values of the weights on the tree are a.s. finite, i.e.,

$$\sum_{n=0}^{\infty} \sum_{\mathbf{i} \in A_n} |Q_{\mathbf{i}}| \Pi_{\mathbf{i}} < \infty \quad \text{a.s.},$$

justifies the following identity

$$R = \sum_{j=1}^{N_\emptyset} C_{(\emptyset,j)} R_j^{(\infty)} + Q_\emptyset = \sum_{j=1}^{N} C_j R_j^{(\infty)} + Q,$$

where $\{R_j^{(\infty)}\}$ are independent copies of R corresponding to the infinite subtree starting with individual j in the first generation. This derivation provides in particular the existence of a solution in distribution to (3).

The set of all solutions to (3) was recently described in [3] (see Theorem 2.3), where it was shown that all solutions can be obtained from the particular explicit solution R given by (9) and a particular nonnegative solution to the fixed-point equation

$$W \stackrel{\mathcal{D}}{=} \sum_{i=1}^{N} C_i^\alpha W_i$$

where $\alpha > 0$ solves $E\left[\sum_{i=1}^N C_i^\alpha\right] = 1$. Nonetheless, from an applications perspective, we are interested in the convergence of the process that results from iterating (3), and we will show that under mild moment conditions on the initial values this procedure always converges to R. Hence, the focus of this paper is only on the tail behavior of R as defined by (9).

As for the solutions to the homogeneous linear equation ($Q \equiv 0$ in (3)), we briefly mention that the set of solutions was fully described in [2], and the power law asymptotics of the particular solution constructed on the weighted branching tree, provided $E\left[\sum_{i=1}^N C_i\right] = 1$, have been previously established in [20,13].

The remainder of the paper is organized as follows. In Section 2.1 we state moment bounds for W_n and R. In Section 2.2 we describe the process that results from iterating the fixed-point equation (3) and show that it converges in distribution to R. The main result for the case where the sum of the weights dominates the behavior of R (the equivalent to the case where N dominates in [14]) is given in Section 3; and the main result for the case where the behavior of R is dominated by Q is given in Section 4. The proofs of the main results are given in Section 5 and some results for weighted random sums, that may be of independent interest, are given in the Appendix.

Notation: Recall that throughout the paper the convention is to denote the random vector associated to the root node $\emptyset$ by $(Q, N, C_1, C_2, \ldots) \equiv (Q_\emptyset, N_\emptyset, C_{(\emptyset,1)}, C_{(\emptyset,2)}, \ldots)$. We will

also use

$$\rho_\beta = E\left[\sum_{i=1}^N C_i^\beta\right] \quad \text{for any } \beta > 0, \text{ and } \rho \equiv \rho_1.$$

2.1. Moments of W_n and R

Let $A_\mathcal{T} = \bigcup_{n=0}^\infty A_n$ and define the operator $x^+ = \max\{x, 0\}$. Now note that

$$W_n^+ \le \sum_{i \in A_n} Q_\mathbf{i}^+ \Pi_\mathbf{i}, \quad n \ge 1,$$

$$\text{and} \quad R^+ \le \sum_{n=0}^\infty W_n^+ \le \sum_{\mathbf{i} \in A_\mathcal{T}} Q_\mathbf{i}^+ \Pi_\mathbf{i},$$

so Lemmas 4.2–4.4 in [16] apply and we immediately obtain the following results; we use $x \vee y = \max\{x, y\}$.

Proposition 2.2. *Assume $E[(Q^+)^\beta] < \infty$. Then,*

1. *if $0 < \beta \le 1$,*

$$E[(W_n^+)^\beta] \le E[(Q^+)^\beta]\rho_\beta^n,$$

2. *if $\beta > 1$, $\rho \vee \rho_\beta < 1$, and $E\left[\left(\sum_{i=1}^N C_i\right)^\beta\right] < \infty$, there exists a finite constant $K_\beta > 0$ such that*

$$E[(W_n^+)^\beta] \le K_\beta(\rho \vee \rho_\beta)^n,$$

for all $n \ge 0$.

Lemma 2.3. *Fix $\beta > 0$ and assume $E[|Q|^\beta] < \infty$. In addition, suppose either* (i) *$\rho_\beta < 1$ for $0 < \beta < 1$, or* (ii) *$\rho \vee \rho_\beta < 1$ and $E\left[\left(\sum_{i=1}^N C_i\right)^\beta\right] < \infty$ for $\beta \ge 1$. Then, $E[|R|^\gamma] < \infty$ for all $0 < \gamma \le \beta$. Moreover, if $\beta \ge 1$, $R^{(n)} \overset{L_\beta}{\to} R$, where L_β stands for convergence in the $(E|\cdot|^\beta)^{1/\beta}$ norm.*

2.2. Iterations of the fixed-point equation

In this section we describe a different process that is more closely related to the ranking of nodes on complex graphs exhibiting a tree-like local behavior. Consider the problem of computing the rank of a randomly chosen node in such graph by using a linear recursion based on the fixed-point equation (3). To be more precise, we will define R_n^* to be the rank of the root node of a tree having n generations and in which the leaf nodes have i.i.d. ranks with the same distribution as some initial condition R_0^*. Then R_n^* can be seen as the rank of the chosen node after n iterations of the algorithm provided that the set of nodes pointing towards it at distance at most n form a tree.

Next we will show that under some technical conditions, this process converges in distribution to R. To this end, given an initial condition R_0^*, define

$$R_n^* \triangleq R^{(n-1)} + W_n(R_0^*), \tag{10}$$

where

$$W_n(R_0^*) = \sum_{\mathbf{i} \in A_n} R_{0,\mathbf{i}}^* \Pi_{\mathbf{i}}$$

and $\{R_{0,\mathbf{i}}^*\}_{\mathbf{i} \in U}$ are i.i.d. copies of R_0^*, independent of the entire weighted branching tree $\mathcal{T}_{Q,C}$. As argued in [16, Section 4.1], this process satisfies

$$R_{n+1}^* \overset{\mathcal{D}}{=} \sum_{i=1}^{N} C_i R_{n,i}^* + Q,$$

where $\{R_{n,i}^*\}_{i \in \mathbb{N}}$ are i.i.d. copies of R_n^*, independent of $(Q, N, C_1, C_2, \dots)$, which corresponds to the process derived by iterating (3) with an initial condition R_0^*. The following lemma shows that $R_n^* \Rightarrow R$ for any initial condition R_0^* satisfying a moment assumption, where $\Rightarrow$ denotes convergence in distribution.

Lemma 2.4. *For any R_0^*, if $E[|Q|^\beta]$, $E[|R_0^*|^\beta] < \infty$ and $\rho_\beta < 1$ for some $0 < \beta \le 1$, then*

$$R_n^* \Rightarrow R,$$

with $E[|R|^\beta] < \infty$. Furthermore, under these assumptions, the distribution of R is the unique solution with finite absolute β-moment to recursion (3).

Proof. In view of (10), and since $R^{(n)} \to R$ a.s., the result will follow from Slutsky's Theorem (see Theorem 25.4 in [5]) once we show that $W_n(R_0^*) \Rightarrow 0$. To this end, recall that $W_n(R_0^*)$ is the same as W_n if we substitute the $Q_\mathbf{i}$ by the $R_{0,\mathbf{i}}^*$. Then, for every $\epsilon > 0$ we have that

$$
\begin{aligned}
P(|W_n(R_0^*)| > \epsilon) &\le \epsilon^{-\beta} E[|W_n(R_0^*)|^\beta] \\
&\le \epsilon^{-\beta} \rho_\beta^n E[|R_0^*|^\beta],
\end{aligned}
$$

where in the second inequality we applied Proposition 2.2(a) to both the positive and negative parts. Since by assumption the right-hand side converges to zero as $n \to \infty$, then $R_n^* \Rightarrow R$. Furthermore, $E[|R|^\beta] < \infty$ by Lemma 2.3. Clearly, under the assumptions, the distribution of R represents the unique solution to (3), since any other possible solution with finite absolute β-moment would have to converge to the same limit. $\square$

In view of Lemma 2.4 and the fact that our ultimate goal is to analyze the asymptotic behavior of $P(R > x)$, in the following sections we will focus on the process $R^{(n)}$ rather than the process R_n^*. This will avoid having to repeat the assumptions on R_0^* throughout the remainder of the paper. Moreover, note that $R_n^* = R^{(n-1)}$ when $R_0^* \equiv 0$, which is a reasonable (simple) initial condition.

3. The case when the sum of the weights dominates

In this section we analyze the tail behavior of $P(R > x)$ as $x \to \infty$ when the sum of the weights,

$$Z_N \triangleq \sum_{i=1}^{N} C_i$$

has a regularly varying distribution. Recall that a function f is regularly varying at infinity with index $-\alpha$, denoted $f \in \mathcal{R}_{-\alpha}$, if $f(x) = x^{-\alpha} L(x)$ for some slowly varying function L; and $L : [0, \infty) \to (0, \infty)$ is slowly varying if $\lim_{x \to \infty} L(\lambda x)/L(x) = 1$ for any $\lambda > 0$.

In this section we focus on the case where $P(Z_N > x) \in \mathcal{R}_{-\alpha}$ for some $\alpha > 1$, and $\rho \vee \rho_\alpha < 1$. The approach that we will follow is similar to that used in [14, Section 5], except for the added complexity of allowing the vector $(Q, N, C_1, C_2, \ldots)$ to be arbitrarily dependent, and allowing non identically distributed weights $\{C_i\}_{i \in \mathbb{N}}$. We start by stating a lemma that describes the asymptotic behavior of $R^{(n)}$. The proof of this lemma is based on the use of some asymptotic limits for randomly stopped and randomly weighted sums recently developed in [22], and adapted to be used in this setting in Theorem A.1 in the Appendix. The main technical difficulty of extending this lemma to steady state ($R = R^{(\infty)}$) is to develop a uniform bound for $R - R^{(n)}$, which is enabled by the main technical result of the paper, Proposition 3.2. The proof of Lemma 3.1 can be found in Section 5.1.

Lemma 3.1. *Let* $Z_N = \sum_{i=1}^{N} C_i$ *and suppose* $\overline{G}(x) = P(Z_N > x) \in \mathcal{R}_{-\alpha}$ *with* $\alpha > 1$, $E[|Q|^{\alpha+\epsilon}] < \infty$, $\rho_{\alpha+\epsilon} < \infty$ *for some* $\epsilon > 0$, $E[Q] > 0$, *and* $\rho < 1$. *Then, for any fixed* $n \in \{1, 2, 3, \ldots\}$,

$$P(R^{(n)} > x) \sim \frac{(E[Q])^\alpha}{(1-\rho)^\alpha} \sum_{k=0}^{n-1} \rho_\alpha^k (1 - \rho^{n-k})^\alpha P(Z_N > x) \tag{11}$$

as $x \to \infty$, *where* $R^{(n)}$ *was defined in* (6).

Remark. We note that for (11) to hold we need to have $E[Q] > 0$, and from the work in [22] it can be seen that if $E[Q] \leq 0$ (which implies $E[R^{(n)}] \leq 0$) we could have $P(R^{(n)} > x) = o(P(Z_N > x))$. In terms of the ranking example given in the introduction, Q usually refers to a nonnegative personalization parameter that determines what page to go to in case the algorithm reaches a page with no outbound links (see [26] for more details).

From Lemma 3.1 one can already guess that, provided $\rho \vee \rho_\alpha < 1$, the tail behavior of R will be

$$P(R > x) \sim \frac{(E[Q])^\alpha}{(1-\rho)^\alpha} \sum_{k=0}^{\infty} \rho_\alpha^k P(Z_N > x)$$

as $x \to \infty$, assuming that the exchange of limits is justified. As mentioned above, this exchange represents the main technical difficulty in the paper (along with its counterpart for the case when Q dominates the behavior of R, discussed in the next section). This result has already been proved in [14] for the case where $Q, N, \{C_i\}$ are all independent, $Q \geq 0$, and the $\{C_i\}$ are i.i.d. using sample-path arguments, and in [26] for the case where (Q, N) is independent of $\{C_i\}$, $Q \geq 0$, the $\{C_i\}$ are i.i.d., using transform methods and Tauberian theorems. Here we follow the approach from [14] where the main tool was a special case of the uniform bound given below. The proof of Proposition 3.2 is given in Section 5.2.

Proposition 3.2. *Let $Z_N = \sum_{i=1}^{N} C_i$ and suppose $\overline{G}(x) = P(Z_N > x) \in \mathcal{R}_{-\alpha}$ with $\alpha > 1$. Assume further that $E[(Q^+)^{\alpha+\epsilon}] < \infty$ and $\rho_{\alpha+\epsilon} < \infty$ for some $\epsilon > 0$. Fix $\rho \vee \rho_\alpha < \eta < 1$. Then, there exists a finite constant $K = K(\eta, \epsilon) > 0$ such that for all $n \geq 1$ and all $x \geq 1$,*

$$P(W_n^+ > x) \leq K\eta^n P(Z_N > x). \tag{12}$$

Remark. Note that we can easily obtain a weaker uniform bound by applying the moment estimate on $E[(W_n^+)^\beta]$ from Proposition 2.2, i.e., $P(W_n^+ > x) \leq E[(W_n^+)^\beta]x^{-\beta} \leq K_\beta(\rho \vee \rho_\beta)^n x^{-\beta}$ for some $0 < \beta < \alpha$, so the tradeoff in (12) is a slightly larger geometric term for a lighter tail distribution.

Proposition 3.2 is the key to establishing that $|R - R^{(n)}|$ goes to zero geometrically fast, which is more precisely stated in the following lemma.

Lemma 3.3. *Let $Z_N = \sum_{i=1}^{N} C_i$ and suppose $\overline{G}(x) = P(Z_N > x) \in \mathcal{R}_{-\alpha}$ with $\alpha > 1$, $E[|Q|^{\alpha+\epsilon}] < \infty$ and $\rho_{\alpha+\epsilon} < \infty$, for some $\epsilon > 0$, and $E[Q] > 0$. Assume $\rho \vee \rho_\alpha < 1$, then, for any fixed $0 < \delta < 1$, $n_0 \in \{1, 2, \ldots\}$ and $\rho \vee \rho_\alpha < \eta < 1$, there exists a finite constant $K > 0$ that does not depend on δ or n_0 such that*

$$\lim_{x \to \infty} \frac{P\left(|R - R^{(n_0)}| > \delta x\right)}{\overline{G}(x)} \leq \frac{K\eta^{n_0+1}}{\delta^{\alpha+1} n_0}.$$

Proof. Fix $\rho \vee \rho_\alpha < \eta_0 < \eta$ and $0 < r < \min\{\alpha - 1, 1\}$. By Potter's Theorem (see Theorem 1.5.6(iii) in [6]), there exists a constant $x_0 = x_0(2, r) \geq 1$ such that for all $x, y \geq x_0$,

$$\frac{\overline{G}(y)}{\overline{G}(x)} \leq 2 \max\left\{(y/x)^{-\alpha+r}, (y/x)^{-\alpha-r}\right\}. \tag{13}$$

Now define $s_{n_0} = \sum_{n=n_0+1}^{\infty} n^{-2} \leq \int_{n_0}^{\infty} t^{-2} dt = n_0^{-1}$, and note that by the union bound,

$$P\left(|R - R^{(n_0)}| > \delta x\right) \leq P\left(\sum_{n=n_0+1}^{\infty} |W_n| > \delta x\right) \leq \sum_{n=n_0+1}^{\infty} P\left(|W_n| > \delta x n^{-2}/s_{n_0}\right).$$

Since in order to be able to use Proposition 3.2 we must have $\delta x n^{-2}/s_{n_0} \geq 1$ we need to split the range of values of n. To this end define $m(x; \delta, n_0) = \lfloor \sqrt{\delta x/(x_0 s_{n_0})} \rfloor$ and apply Proposition 3.2 to the terms $n_0 + 1 \leq n \leq m(x; \delta, n_0)$ (to both the positive and negative parts of W_n), and Markov's inequality to the terms $n > m(x; \delta, n_0)$, to obtain

$$\sum_{n=n_0+1}^{\infty} P\left(|W_n| > \delta x n^{-2}/s_{n_0}\right) \leq \sum_{n=n_0+1}^{m(x;\delta,n_0)} K_0 \eta_0^n \overline{G}(\delta x n^{-2}/s_{n_0})$$

$$+ \sum_{n=m(x;\delta,n_0)+1}^{\infty} \frac{E[|W_n|^{\alpha-r}]}{(\delta x n^{-2}/s_{n_0})^{\alpha-r}}$$

$$\leq \sum_{n=n_0+1}^{m(x;\delta,n_0)} 2K_0 \eta_0^n \left(\frac{\delta x n^{-2}/s_{n_0}}{x}\right)^{-\alpha-r} \overline{G}(x)$$

$$+ \sum_{n=m(x;\delta,n_0)+1}^{\infty} \frac{K_{\alpha-r}(\rho \vee \rho_{\alpha-r})^n}{(\delta x n^{-2}/s_{n_0})^{\alpha-r}},$$

where in the second inequality we used (13) and Proposition 2.2, and $K_0 = K_0(\eta_0, \epsilon)$. It follows that

$$
P\left(|R - R^{(n_0)}| > \delta x\right) \leq \frac{2K_0(s_{n_0})^{\alpha+r}}{\delta^{\alpha+r}} \sum_{n=n_0+1}^{m(x;\delta,n_0)} \eta_0^n n^{2(\alpha+r)} \overline{G}(x)
$$
$$
+ \frac{K_{\alpha-r}(s_{n_0})^{\alpha-r}}{\delta^{\alpha-r}} \sum_{n=m(x;\delta,n_0)+1}^{\infty} \frac{(\rho \vee \rho_{\alpha-r})^n n^{2(\alpha-r)}}{x^{\alpha-r}}.
$$

Now note that by the convexity of $f(\theta) = \rho_\theta$ we have $\rho \vee \rho_{\alpha-r} \leq \rho \vee \rho_\alpha < \eta_0$, from where it follows that

$$
P\left(|R - R^{(n_0)}| > \delta x\right) \leq \frac{(2K_0 + K_{\alpha-r})(s_{n_0})^{\alpha-r}}{\delta^{\alpha+r}} \left(\sum_{n=n_0+1}^{\infty} \eta_0^n n^{2(\alpha+r)} \overline{G}(x) \right.
$$
$$
+ \left. \sum_{n=m(x;\delta,n_0)+1}^{\infty} \eta_0^n n^{2(\alpha+r)} x^{-\alpha+r} \right)
$$
$$
\leq \frac{2K_0 + K_{\alpha-r}}{\delta^{\alpha+r} n_0^{\alpha-r}(1-\eta)} \cdot \sup_{m\geq 1}(\eta_0/\eta)^m m^{2(\alpha+r)}
$$
$$
\times \left(\sum_{n=n_0+1}^{\infty} (1-\eta)\eta^n \overline{G}(x) + \sum_{n=m(x;\delta,n_0)+1}^{\infty} (1-\eta)\eta^n x^{-\alpha+r} \right)
$$
$$
\triangleq \frac{K}{\delta^{\alpha+r} n_0^{\alpha-r}} \left(\eta^{n_0+1}\overline{G}(x) + \eta^{m(x;\delta,n_0)+1} x^{-\alpha+r} \right),
$$

where $K = K(\eta_0, \epsilon, \eta, r)$ does not depend on δ or n_0. It follows that for $\overline{G}(x) = x^{-\alpha} L(x)$,

$$
\lim_{x\to\infty} \frac{P\left(R - R^{(n_0)} > \delta x\right)}{\overline{G}(x)} \leq \frac{K\eta^{n_0+1}}{\delta^{\alpha+1} n_0} \left(1 + \lim_{x\to\infty} \frac{\eta^{\sqrt{\delta x/(x_0 s_{n_0})}-n_0-1} x^r}{L(x)} \right)
$$
$$
= \frac{K\eta^{n_0+1}}{\delta^{\alpha+1} n_0}. \quad \square
$$

Having stated Lemma 3.3, we can now prove the main theorem of this section.

Theorem 3.4. *Let $Z_N = \sum_{i=1}^{N} C_i$ and suppose $\overline{G}(x) = P(Z_N > x) \in \mathcal{R}_{-\alpha}$ with $\alpha > 1$, $E[|Q|^{\alpha+\epsilon}] < \infty$ and $\rho_{\alpha+\epsilon} < \infty$, for some $\epsilon > 0$, and $E[Q] > 0$. Assume $\rho \vee \rho_\alpha < 1$, then,*

$$
P(R > x) \sim \frac{(E[Q])^\alpha}{(1-\rho)^\alpha (1-\rho_\alpha)} P(Z_N > x)
$$

as $x \to \infty$, where R was defined in (9).

Remarks. (i) For the case where the $\{C_i\}$ are i.i.d. and independent of N, and $P(N > x) \in \mathcal{R}_{-\alpha}$, Lemma 3.7(2) in [17] gives

$$
P(Z_N > x) \sim (E[C_1])^\alpha P(N > x) \quad \text{as } x \to \infty.
$$

(ii) Given the previous remark, it follows that Theorem 3.4 generalizes both Theorem 5.1 in [14] (for Q, N, $\{C_i\}$ all independent, $Q \geq 0$ and $\{C_i\}$ i.i.d.) and the corresponding result from Section 3.4 in [26] (for (Q, N) independent of $\{C_i\}$, $Q \geq 0$, $\{C_i\}$ i.i.d., $E[Q] < 1$ and $E[C] = (1 - E[Q])/E[N]$).

(iii) In view of Lemma 3.1, the theorem shows that the limits $\lim_{x\to\infty} \lim_{n\to\infty} P(R^{(n)} > x)/P(N > x)$ are interchangeable.

Proof of Theorem 3.4. Fix $0 < \delta < 1$ and $n_0 \geq 1$. Choose $\rho \vee \rho_\alpha < \eta < 1$ and use Proposition 3.2 to obtain that for some constant $K_0 > 0$,

$$P(W_n^+ > x) \leq K_0 \eta^n P(Z_N > x)$$

for all $n \geq 1$ and all $x \geq 1$. Let $H_n = (E[Q])^\alpha (1 - \rho)^{-\alpha} \sum_{k=0}^{n-1} \rho_\alpha^k (1 - \rho^{n-k})^\alpha$ and $H = H_\infty$. Then,

$$|P(R > x) - H P(Z_N > x)| \leq \left| P(R > x) - P(R^{(n_0)} > x) \right| \tag{14}$$

$$+ \left| P(R^{(n_0)} > x) - H_{n_0} P(Z_N > x) \right| \tag{15}$$

$$+ \left| H_{n_0} - H \right| P(Z_N > x). \tag{16}$$

By Lemma 3.1, there exists a function $\varphi(x) \downarrow 0$ as $x \to \infty$ such that

$$\left| P(R^{(n_0)} > x) - H_{n_0} P(Z_N > x) \right| \leq \varphi(x) H P(Z_N > x), \tag{17}$$

which can be used to bound (15). Next, for (16) simply note that

$$
\begin{aligned}
\frac{1}{H} \left| H_{n_0} - H \right| &= (1 - \rho_\alpha) \left(\sum_{k=0}^{\infty} \rho_\alpha^k - \sum_{k=0}^{n_0-1} \rho_\alpha^k (1 - \rho^{n_0-k})^\alpha \right) \\
&= (1 - \rho_\alpha) \sum_{k=0}^{n_0-1} \rho_\alpha^k (1 - (1 - \rho^{n_0-k})^\alpha) + (1 - \rho_\alpha) \sum_{k=n_0}^{\infty} \rho_\alpha^k \\
&\leq (1 - \rho_\alpha) \sum_{k=0}^{n_0-1} \rho_\alpha^k \alpha \rho^{n_0-k} + \rho_\alpha^{n_0} \\
&\leq (\alpha(1 - \rho_\alpha)n_0 + 1)(\rho \vee \rho_\alpha)^{n_0} \\
&\leq \left(\alpha \sup_{m \geq 1} \left(\frac{\rho_\alpha \vee \rho}{\eta} \right)^m m \right) \eta^{n_0} \triangleq K' \eta^{n_0}.
\end{aligned}
$$

The rest of the proof is basically an analysis of (14). We start by noting that

$$\left| P(R > x) - P(R^{(n_0)} > x) \right| \leq \left| P\left(R > x, |R - R^{(n_0)}| \leq \delta x \right) - P\left(R^{(n_0)} > x \right) \right|$$
$$+ P\left(R > x, |R - R^{(n_0)}| > \delta x \right).$$

Also, since

$$P\left(R^{(n_0)} > (1 + \delta)x, |R - R^{(n_0)}| \leq \delta x \right) \leq P\left(R > x, |R - R^{(n_0)}| \leq \delta x \right)$$
$$\leq P\left(R^{(n_0)} > (1 - \delta)x \right)$$

and if $\overline{a} \leq a \leq \underline{a}$, then $|a - b| \leq |\underline{a} - b| + |\overline{a} - b|$, we have

$$\left| P(R > x) - P(R^{(n_0)} > x) \right|$$

$$\leq \left| P\left(R^{(n_0)} > (1+\delta)x, |R - R^{(n_0)}| \leq \delta x\right) - P\left(R^{(n_0)} > x\right) \right|$$

$$+ \left| P\left(R^{(n_0)} > (1-\delta)x\right) - P\left(R^{(n_0)} > x\right) \right| + P\left(R > x, |R - R^{(n_0)}| > \delta x\right)$$

$$= P\left(R^{(n_0)} > (1-\delta)x\right) - P\left(R^{(n_0)} > (1+\delta)x, |R - R^{(n_0)}| \leq \delta x\right)$$

$$+ P\left(R > x, |R - R^{(n_0)}| > \delta x\right)$$

$$\leq P\left(R^{(n_0)} > (1-\delta)x\right) - P\left(R^{(n_0)} > (1+\delta)x\right) \tag{18}$$

$$+ 2P\left(|R - R^{(n_0)}| > \delta x\right). \tag{19}$$

From (17) and the observation that $H_{n_0} \leq H$, it follows that (18) is bounded by

$$P(R^{(n_0)} > (1-\delta)x) - H_{n_0} P(Z_N > (1-\delta)x)$$
$$+ H_{n_0} \left(P(Z_N > (1-\delta)x) - P(Z_N > (1+\delta)x)\right)$$
$$+ H_{n_0} P(Z_N > (1+\delta)x) - P(R^{(n_0)} > (1+\delta)x)$$

$$\leq \left\{ 2\varphi((1-\delta)x)\frac{\overline{G}((1-\delta)x)}{\overline{G}(x)} + \left(\frac{\overline{G}((1-\delta)x)}{\overline{G}(x)} - \frac{\overline{G}((1+\delta)x)}{\overline{G}(x)}\right)\right\} H P(Z_N > x).$$

Moreover, since $\overline{G} \in \mathcal{R}_{-\alpha}$ and $\varphi((1-\delta)x) \to 0$, then

$$2\varphi((1-\delta)x)\frac{\overline{G}((1-\delta)x)}{\overline{G}(x)} + \left(\frac{\overline{G}((1-\delta)x)}{\overline{G}(x)} - \frac{\overline{G}((1+\delta)x)}{\overline{G}(x)}\right)$$

$$\to (1-\delta)^{-\alpha} - (1+\delta)^{-\alpha}$$

as $x \to \infty$. To analyze (19) use Lemma 3.3 to obtain

$$\lim_{x \to \infty} \frac{2P\left(|R - R^{(n_0)}| > \delta x\right)}{H P(Z_N > x)} \leq \frac{K''\eta^{n_0+1}}{\delta^{\alpha+1} n_0}$$

for any $\rho \vee \rho_\alpha < \eta < 1$ and some constant $K'' > 0$ that does not depend on δ or n_0.

Finally, by replacing the preceding estimates in (14)–(16), we obtain

$$\lim_{x \to \infty} \left| \frac{P(R > x)}{H P(Z_N > x)} - 1 \right| \leq (1-\delta)^{-\alpha} - (1+\delta)^{-\alpha} + \frac{K\eta^{n_0}}{\delta^{\alpha+1}}.$$

Since the right hand side can be made arbitrarily small by first letting $n_0 \to \infty$ and then $\delta \downarrow 0$, the result of the theorem follows. $\square$

4. The case when Q dominates

This section of the paper treats the case when the heavy-tailed behavior of R arises from the $\{Q_i\}$, known in the autoregressive processes literature as innovations. This setting is well known in the special case $N \equiv 1$, since then the linear fixed-point equation (3) reduces to

$$R \stackrel{\mathcal{D}}{=} CR + Q,$$

where (C, Q) are generally dependent. This fixed-point equation is the one satisfied by the steady state of the autoregressive process of order one with random coefficients, RCA(1) (see [18,7,10, 12]). The power law asymptotics of the solution R in this context were established in the classical work by Kesten [18] (multivariate setting), and through implicit renewal theory in the paper by Goldie [10]. In both of these works the assumptions include the existence of an $\alpha > 0$ such that $E[|C|^\alpha] = 1$, $E[|C|^\alpha \log^+ |C|] < \infty$, and $E[|Q|^\alpha] < \infty$.

That the innovations $\{Q_i\}$ can give raise to heavy tails when the α mentioned above does not exist is also well known, see, e.g. [12,19]; the main theorem of this section provides an alternative derivation of the forward implication in Theorem 1 from [12] (see also Proposition 2.4 in [19]) in the more general context of $N \geq 0$. We also mention that Theorem 1 in [12] includes the case where $\alpha \in (0, 1]$, which would require a different proof technique from the one in this paper.

The results presented here are very similar to those in Section 3, and so are their proofs. We will therefore only present the statements and skip most of the proofs. We start with the equivalent of Lemma 3.1 in this context; its proof can be found in Section 5.1.

Lemma 4.1. *Suppose $P(Q > x) \in \mathcal{R}_{-\alpha}$, with $\alpha > 1$, $E[(Q^-)^{1+\epsilon}] < \infty$ and $E[Z_N^{\alpha+\epsilon}] < \infty$, for some $\epsilon > 0$. Then, for any fixed $n \in \{1, 2, 3, \ldots\}$,*

$$P(R^{(n)} > x) \sim \sum_{k=0}^{n} \rho_\alpha^k P(Q > x)$$

as $x \to \infty$, where $R^{(n)}$ was defined in (9).

As for the case when $Z_N = \sum_{i=1}^{N} C_i$ dominates the asymptotic behavior of R, we can expect that,

$$P(R > x) \sim (1 - \rho_\alpha)^{-1} P(Q > x),$$

and the technical difficulty is justifying the exchange of limits. The same techniques used in Section 3 can be used in this case as well. The corresponding version of Proposition 3.2 is given below. We point out that even though the condition $\rho < 1$ is not necessary for the proportionality constant in Lemma 4.1 to be finite, it is required for the finiteness of $E[|R|]$, so it is natural that it appears as part of the hypothesis in all the other results in this section.

Proposition 4.2. *Suppose $P(Q > x) \in \mathcal{R}_{-\alpha}$, with $\alpha > 1$, $E[Z_N^{\alpha+\epsilon}] < \infty$ for some $\epsilon > 0$, and let $\rho \vee \rho_\alpha < \eta < 1$. Then, there exists a constant $K = K(\eta, \epsilon) > 0$ such that for all $n \geq 1$ and all $x \geq 1$,*

$$P(W_n > x) \leq K \eta^n P(Q > x).$$

A sketch of the proof can be found in Section 5.2.

The corresponding version of Lemma 3.3 is given below. Its proof is basically identical to that of Lemma 3.3 and is therefore omitted.

Lemma 4.3. *Let $Z_N = \sum_{i=1}^{N} C_i$ and suppose $P(Q > x) \in \mathcal{R}_{-\alpha}$ with $\alpha > 1$, $E[Z_N^{\alpha+\epsilon}] < \infty$ for some $\epsilon > 0$, and $E[|Q|^\beta] < \infty$ for all $0 < \beta < \alpha$. Assume $\rho \vee \rho_\alpha < 1$, then, for any fixed $0 < \delta < 1$, $n_0 \in \{1, 2, \ldots\}$ and $\rho \vee \rho_\alpha < \eta < 1$, there exists a constant $K > 0$ that does not depend on δ or n_0 such that*

$$\lim_{x \to \infty} \frac{P\left(|R - R^{(n_0)}| > \delta x\right)}{P(Q > x)} \leq \frac{K \eta^{n_0+1}}{\delta^{\alpha+1} n_0}.$$

And finally, the main theorem of this section. The proof again greatly resembles that of Theorem 3.4 and is therefore omitted.

Theorem 4.4. *Suppose $P(Q > x) \in \mathcal{R}_{-\alpha}$, with $\alpha > 1$, $E[|Q|^\beta] < \infty$ for all $0 < \beta < \alpha$. Assume $\rho \vee \rho_\alpha < 1$, and $E[Z_N^{\alpha+\epsilon}] < \infty$ for some $\epsilon > 0$. Then,*

$$P(R > x) \sim (1 - \rho_\alpha)^{-1} P(Q > x)$$

as $x \to \infty$, where R was defined in (9).

Remarks. (i) This result generalizes Theorem 1 in [12] for the case $N \equiv 1$ (the forward implication, $\alpha > 1$) to the weighted branching tree. It also generalizes the results in [26,14] in the same way as Theorem 3.4 does for the case where Z_N dominates.
(ii) It is also worth pointing out that the same sample path techniques used here can be used to study the intermediate case where $P(Q > x) \sim K P(Z_N > x)$ for some constant $K > 0$, which is also analyzed in [26] under stronger conditions than those in this paper.

5. Proofs

5.1. Finite iterations of $R^{(n)}$

This section contains the proofs of Lemmas 3.1 and 4.1, which refer to the asymptotic behavior of $P(R^{(n)} > x)$ for any finite n.

Proof of Lemma 3.1. We proceed by induction in n. For $n = 1$ fix $\alpha/(\alpha + \epsilon) < \delta < 1$ and note that

$$P(R^{(1)} > x) = P\left(\sum_{i=1}^{N_\emptyset} C_{(\emptyset,i)} Q_{(\emptyset,i)} + Q_\emptyset > x\right)$$

$$= P\left(\sum_{i=1}^{N} C_i Q_i > x - Q, |Q| \leq x^\delta\right)$$

$$+ P\left(\sum_{i=1}^{N} C_i Q_i > x - Q, |Q| > x^\delta\right),$$

where Q is independent of the $\{Q_i\}$ but not of $(N, C_1, C_2, \ldots)$. By Theorem 2.6 in [22] and the regular variation of $\overline{G}$,

$$P\left(\sum_{i=1}^{N} C_i Q_i > x - Q, |Q| \leq x^\delta\right) \leq P\left(\sum_{i=1}^{N} C_i Q_i > x - x^\delta\right)$$

$$\sim P\left(Z_N > (x - x^\delta)/E[Q]\right) \sim (E[Q])^\alpha \overline{G}(x),$$

as $x \to \infty$, and

$$P\left(\sum_{i=1}^{N} C_i Q_i > x - Q, |Q| \leq x^\delta\right) \geq P\left(\sum_{i=1}^{N} C_i Q_i > x + x^\delta\right) - P\left(|Q| > x^\delta\right)$$

$$= (E[Q])^\alpha \overline{G}(x)(1 + o(1)) - P\left(|Q| > x^\delta\right).$$

Now note that by Markov's inequality,

$$P\left(|Q| > x^\delta\right) \leq \frac{E[|Q|^{\alpha+\epsilon}]}{x^{\delta(\alpha+\epsilon)}} = o\left(\overline{G}(x)\right)$$

as $x \to \infty$. Therefore,

$$P(R^{(1)} > x) \sim (E[Q])^\alpha \overline{G}(x).$$

Now suppose that we have

$$P(R^{(n)} > x) \sim \frac{(E[Q])^\alpha}{(1-\rho)^\alpha} \sum_{k=0}^{n-1} \rho_\alpha^k (1 - \rho^{n-k})^\alpha \overline{G}(x) \triangleq H_n \overline{G}(x).$$

By [Theorem A.1](#),

$$\begin{aligned}
P(R^{(n+1)} > x) = P\left(\sum_{i=1}^{N} C_i R_i^{(n)} + Q > x\right) \\
\sim \left(\rho_\alpha + H_n^{-1}(E[R^{(n)}])^\alpha\right) P(R^{(n)} > x) \\
\sim \left(H_n \rho_\alpha + (E[R^{(n)}])^\alpha\right) \overline{G}(x).
\end{aligned}$$

Next, observing that $E[R^{(n)}] = \sum_{i=0}^{n} E[W_i] = E[Q] \sum_{i=0}^{n} \rho^i = E[Q](1 - \rho^{n+1})/(1 - \rho)$, we obtain

$$\begin{aligned}
H_n \rho_\alpha + (E[R^{(n)}])^\alpha &= H_n \rho_\alpha + (E[Q])^\alpha \frac{(1 - \rho^{n+1})^\alpha}{(1-\rho)^\alpha} \\
&= \frac{(E[Q])^\alpha}{(1-\rho)^\alpha} \sum_{k=0}^{n-1} \rho_\alpha^{k+1} (1 - \rho^{n-k})^\alpha + \frac{(E[Q])^\alpha}{(1-\rho)^\alpha}(1 - \rho^{n+1})^\alpha \\
&= \frac{(E[Q])^\alpha}{(1-\rho)^\alpha} \sum_{j=1}^{n} \rho_\alpha^{j} (1 - \rho^{n+1-j})^\alpha + \frac{(E[Q])^\alpha}{(1-\rho)^\alpha}(1 - \rho^{n+1})^\alpha \\
&= H_{n+1}.
\end{aligned}$$

This completes the proof. $\quad\square$

Proof of Lemma 4.1. We proceed by induction in n. By [Theorem A.2](#),

$$P(R^{(1)} > x) = P\left(\sum_{i=1}^{N_\emptyset} C_{(\emptyset,i)} Q_{(\emptyset,i)} + Q_\emptyset > x\right) \sim (\rho_\alpha + 1) P(Q > x)$$

as $x \to \infty$.

Now suppose that we have

$$P(R^{(n)} > x) \sim \sum_{k=0}^{n} \rho_\alpha^k P(Q > x) \triangleq H_n P(Q > x).$$

Then by [Theorem A.2](#) again

$$P(R^{(n+1)} > x) = P\left(\sum_{i=1}^{N_\emptyset} C_{(\emptyset,i)} R_i^{(n)} + Q_\emptyset > x\right)$$

$$\sim \left(\rho_\alpha + H_n^{-1}\right) P(R^{(n)} > x)$$
$$\sim (\rho_\alpha H_n + 1) P(Q > x)$$
$$= H_{n+1} P(Q > x). \quad \square$$

5.2. Uniform bounds for $P(W_n > x)$

This section contains the proof of Proposition 3.2 and a sketch of the proof of Proposition 4.2. The first proof is rather involved, and a great effort goes into obtaining a bound for one iteration of the recursion satisfied by W_n, so for the convenience of the reader it is presented separately in Lemma 5.1. This lemma can also be used to prove the corresponding uniform bound for W_n in the case when Q dominates the behavior of R. Throughout this section assume that $1/L(x)$ is locally bounded on $[1, \infty)$, and recall that if $L(t)$ is slowly varying, then $\lim_{t \to \infty} t^\varepsilon L(t) = \infty$ for any $\varepsilon > 0$.

Lemma 5.1. *Suppose that $P(Z_N > x) \leq x^{-\alpha} L(x)$, with $\alpha > 1$ and $L(\cdot)$ slowly varying, $\rho \vee \rho_\alpha < 1$, and $\rho_{\alpha+\epsilon} < \infty$ for some $\epsilon > 0$. Assume further that $E[(Q^+)^\beta] < \infty$ for any $0 < \beta < \alpha$. Then, for any $0 < \delta < \min\{(\alpha - 1)/2, \epsilon, 1/2\}$ and any $T > 0$, there exists a finite constant $K = K(\epsilon, \delta, T) > 0$ that does not depend on n, such that for all $0 \leq n \leq \frac{\epsilon}{2|\log(\rho \vee \rho_\alpha)|} \log x$ and all $x \geq 1$,*

$$P(W_{n+1}^+ > x) \leq K(\rho \vee \rho_\alpha)^n x^{-\alpha} L(x)$$
$$+ E\left[1\left(\sup_{1 \leq i < N+1} C_i \leq x/T \right) \sum_{i=1}^N 1\left(C_i W_{n,i}^+ > (1 - \delta)x \right) \right].$$

To ease the reading of the proof of Lemma 5.1 we will split it further into several lemmas. To avoid repetition we give below most of the notation that will be used. We start by defining

$$I_N(t) = \#\{1 \leq i < N+1 : C_i > t\}$$
$$J_N(t) = \#\{1 \leq i < N+1 : C_i W_{n,i}^+ > t\}.$$

When $N = \infty$ the convention will be to interpret $N + 1 = N = \infty$; the use of the range $1 \leq i < N + 1$ then guarantees that i is finite.

For the same $\epsilon > 0$ and $0 < \delta < \min\{(\alpha - 1)/2, \epsilon, 1/2\}$ from the statement of the lemma, define $\gamma_n = \|W_n^+\|_{1+\delta} = (E[(W_n^+)^{1+\delta}])^{1/(1+\delta)}$. We also define

$$\nu = \epsilon/(2(\alpha + \epsilon)), \qquad y = x/\log x, \qquad w = x^{1-\nu},$$
$$a_n = \delta^{-2} E[Q^+](\rho \vee \rho_{1+\delta})^{n/(1+\delta)}.$$

Before going into the proof, we would like to emphasize that special care goes into making sure that K in the statement of the lemma does not depend on n. This is important since Lemma 5.1 will be applied iteratively in the proof of Proposition 3.2, where one does not want K to grow from one iteration to the next.

Lemma 5.2. *Under the assumptions of Lemma 5.1, there exists a constant $K_1 = K_1(\delta, \nu, c)$ such that for all $x \geq 1$ and all $0 \leq n \leq c \log x$,*

$$P\left(\sum_{i=1}^N C_i W_{n,i}^+ 1(C_i W_{n,i}^+ \leq y) > \delta x, Z_N \leq x/a_n, I_N(w/\gamma_n) = 0 \right)$$
$$\leq K_1(\rho \vee \rho_\alpha)^n x^{-\alpha} L(x).$$

Proof. By Lemma 3.2 in [22] (with $v = y$, $u = w$, $z = \delta x$, $\eta = 1 + \delta$, and $A = (-\infty, x/a_n]$), there exists a constant $K_{1,1} = K_{1,1}(\delta) > 1$ such that (25) is bounded by

$$E\left[1(Z_N \leq x/a_n) e^{-\frac{\delta}{y} \log(y/w) \left(\delta x - \left(E[W_n^+] + \frac{K_{1,1} \|W_n^+\|_{1+\delta}}{\log(y/w)} \right)^+ Z_N \right)} \right]$$

$$\leq e^{-\frac{\delta}{y} \log(y/w) \left(\delta x - \left(E[W_n^+] + \frac{K_{1,1} \|W_n^+\|_{1+\delta}}{\log(y/w)} \right)^+ x/a_n \right)}$$

$$\leq e^{-\frac{\delta}{y} \log(y/w) \left(\delta x - \left(E[Q^+]\rho^n + \frac{K_{1,1}\gamma_n}{\log(y/w)} \right) x/a_n \right)},$$

where we used $E[W_n^+] \leq E[Q^+]\rho^n$ and $\gamma_n = \|W_n^+\|_{1+\delta}$. From Proposition 2.2 we know that $\|W_n^+\|_{1+\delta} = \left(E[(W_n^+)^{1+\delta}] \right)^{1/(1+\delta)} \leq K_{1,2}(\rho \vee \rho_{1+\delta})^{n/(1+\delta)} \leq K_{1,2}(\delta^2/E[Q^+])a_n$, where $K_{1,2} = K_{1,2}(\delta) > 0$ is a finite constant. It follows that (25) is bounded by

$$e^{-\frac{\delta}{y} \log(y/w) \left(\delta x - \left(\delta^2 (\rho \vee \rho_{1+\delta})^{n\delta/(1+\delta)} + \frac{K_{1,1}K_{1,2}\delta^2}{E[Q^+]\log(y/w)} \right) x \right)}$$

$$\leq e^{-\delta^2 \log x \log(y/w) \left(1 - \delta - \frac{K_{1,1}K_{1,2}\delta}{E[Q^+]\log(y/w)} \right)}$$

$$= e^{-\delta^2 v (\log x)^2 \left(1 - \frac{\log\log x}{v \log x} \right) \left(1 - \delta - \frac{K_{1,1}K_{1,2}\delta}{E[Q^+]\log(x^v/\log x)} \right)}$$

$$\leq e^{-\delta^2 v (1-2\delta)^2 (\log x)^2}, \tag{20}$$

where the last inequality holds for all $x \geq x_1$ for some $x_1 = x_1(\delta, v) > 0$. Now we choose $x_2 = x_2(\delta, v, c) \geq x_1$ such that $\delta^2 v(1 - 2\delta)^2 \log x \geq \alpha + \delta + c|\log(\rho \vee \rho_\alpha)|$ for all $x \geq x_2$ to obtain that (20) is bounded by

$$e^{-(\alpha+\delta)\log x - c|\log(\rho \vee \rho_\alpha)|\log x} = \frac{1}{x^{\alpha+\delta}}(\rho \vee \rho_\alpha)^{c\log x},$$

for all $x \geq x_2$. Next, define $K_{1,3} = K_{1,3}(\delta, v, c)$ as

$$K_{1,3} = \sup_{1 \leq t \leq x_2} \frac{1}{t^{-\alpha-\delta}(\rho \vee \rho_\alpha)^{c\log t}} \cdot e^{-\delta^2 v(\log t)^2 \left(1 - \frac{\log\log t}{v \log t} \right) \left(1 - \delta - \frac{K_{1,1}K_{1,2}\delta}{E[Q^+]\log(t^v/\log t)} \right)} < \infty,$$

to obtain that

$$P\left(\sum_{i=1}^N C_i W_{n,i}^+ 1(C_i W_{n,i}^+ \leq y) > \delta x, Z_N \leq x/a_n, I_N(w/\gamma_n) = 0 \right)$$

$$\leq \frac{K_{1,3}}{x^{\alpha+\delta}}(\rho \vee \rho_\alpha)^{c\log x}$$

$$\leq \frac{K_{1,3}}{x^{\alpha+\delta}}(\rho \vee \rho_\alpha)^n$$

for all $x \geq 1$ and $0 \leq n \leq c\log x$. Finally, set $K_1 = K_{1,3}\sup_{t \geq 1}(t^\delta L(t))^{-1}$ to complete the proof. $\square$

Lemma 5.3. *Under the assumptions of Lemma 5.1, there exists a finite constant $K_2 = K_2(\delta, v)$ such that for all $x \geq 1$,*

$$P(J_N(y) \geq 2, Z_N \leq x/a_n, I_N(w/\gamma_n) = 0) \leq K_2(\rho \vee \rho_\alpha)^n x^{-\alpha} L(x).$$

Proof. Let $\mathcal{F} = \sigma(N, C_1, C_2, \ldots)$ and use the union bound to obtain

$$P\left(J_N(y) \geq 2, Z_N \leq x/a_n, I_N(w/\gamma_n) = 0\right)$$

$$= E\Bigg[1(Z_N \leq x/a_n, I_N(w/\gamma_n) = 0)$$

$$\times E\left[1\left(\bigcup_{1 \leq i < j < N+1} \{C_i W_{n,i}^+ > y, C_j W_{n,j}^+ > y\} \right) \bigg| \mathcal{F} \right] \Bigg]$$

$$\leq E\Big[1(Z_N \leq x/a_n, I_N(w/\gamma_n) = 0)$$

$$\times \sum_{1 \leq i < j < N+1} E\left[1\left(C_i W_{n,i}^+ > y, C_j W_{n,j}^+ > y \right) \bigg| \mathcal{F} \right] \Big]$$

$$\leq E\Bigg[1(Z_N \leq x/a_n, I_N(w/\gamma_n) = 0) \left(\sum_{i=1}^{N} E\left[1\left(C_i W_{n,i}^+ > y \right) \bigg| C_i \right] \right)^2 \Bigg],$$

where in the last step we used the conditional independence of $C_i W_{n,i}^+$ and $C_j W_{n,j}^+$ given $\mathcal{F}$. Now, by Markov's inequality,

$$1(I_N(w/\gamma_n) = 0) \sum_{i=1}^{N} E\left[1\left(C_i W_{n,i}^+ > y \right) \bigg| C_i \right] \leq 1(I_N(w/\gamma_n) = 0)$$

$$\times \sum_{i=1}^{N} \frac{E[(C_i W_{n,i}^+)^{1+\delta} | C_i]}{y^{1+\delta}}$$

$$= \frac{1}{y^{1+\delta}} 1(I_N(w/\gamma_n) = 0) \sum_{i=1}^{N} C_i^{1+\delta} \gamma_n^{1+\delta}$$

$$\leq \frac{1}{y^{1+\delta}} (w/\gamma_n)^{\delta} \gamma_n^{1+\delta} Z_N = \frac{w^{\delta}}{y^{1+\delta}} \gamma_n Z_N.$$

Similarly, for $\beta = \alpha - \delta\nu/2 > 1$,

$$\sum_{i=1}^{N} E\left[1\left(C_i W_{n,i}^+ > y \right) \bigg| C_i \right] \leq \sum_{i=1}^{N} \frac{E[(C_i W_{n,i}^+)^{\beta} | C_i]}{y^{\beta}} \leq \frac{E[(W_n^+)^{\beta}]}{y^{\beta}} \sum_{i=1}^{N} C_i^{\beta}.$$

It follows that

$$P\left(J_N(y) \geq 2, Z_N \leq x/a_n, I_N(w/\gamma_n) = 0\right) \leq E\Bigg[1(Z_N \leq x/a_n) Z_N \sum_{i=1}^{N} C_i^{\beta} \Bigg]$$

$$\times \frac{w^{\delta}}{y^{\beta+1+\delta}} \gamma_n E[(W_n^+)^{\beta}]$$

$$\leq E\Bigg[\sum_{i=1}^{N} C_i^{\beta} \Bigg] \frac{w^{\delta} x}{y^{\beta+1+\delta}} \cdot \frac{\gamma_n}{a_n} E[(W_n^+)^{\beta}]$$

$$= \rho_{\beta} \frac{(\log x)^{\beta+1+\delta}}{x^{\alpha+\delta\nu/2}} \cdot \frac{\gamma_n}{a_n} E[(W_n^+)^{\beta}]. \qquad (21)$$

By Proposition 2.2, there exists a constant $K_{2,1} = K_{2,1}(\beta) > 0$ such that $E[(W_n^+)^\beta] \leq K_{2,1}(\rho \vee \rho_\beta)^n$. This combined with the observation $\gamma_n \leq K_{1,2}(\delta^2/E[Q^+])a_n$ for some constant $K_{1,2} = K_{1,2}(\delta)$ made in the proof of Lemma 5.2 gives that (21) is bounded by

$$
\left(\rho_\beta \cdot \frac{K_{1,2}K_{2,1}\delta^2}{E[Q^+]} \cdot \frac{(\log x)^{\beta+1+\delta}}{x^{\delta v/2}L(x)} \right) (\rho \vee \rho_\beta)^n x^{-\alpha}L(x)
$$

$$
\leq \left(\rho_\beta \cdot \frac{K_{1,2}K_{2,1}\delta^2}{E[Q^+]} \cdot \sup_{t\geq 1} \frac{(\log t)^{\beta+1+\delta}}{t^{\delta v/2}L(t)} \right) (\rho \vee \rho_\beta)^n x^{-\alpha}L(x)
$$

for all $x \geq 1$. The convexity of $f(\theta) = \rho_\theta$ gives $\rho \vee \rho_\beta \leq \rho \vee \rho_\alpha$, which completes the proof.   □

Lemma 5.4. *Under the assumptions of Lemma 5.1, there exists a finite constant $K_3 = K_3(\delta, \epsilon, T)$ such that for all $x \geq 1$,*

$$
P(I_N(w/\gamma_n) \geq 1) + P(I_N(w/\gamma_n) = 0, I_N(x/T) \geq 1) \leq K_3 \left((\rho \vee \rho_\alpha)^n + \frac{1}{x^{\epsilon/2}} \right)
$$

$$
\times x^{-\alpha}L(x).
$$

Proof. First note that from the union bound we obtain

$$
P(I_N(w/\gamma_n) \geq 1) = E\left[1\left(\bigcup_{i=1}^N \{C_i > w/\gamma_n\} \right) \right] \leq E\left[\sum_{i=1}^N 1(C_i > w/\gamma_n) \right]
$$

$$
= \sum_{i=1}^\infty P(C_i > w/\gamma_n, N \geq i)
$$

$$
= \sum_{i=1}^\infty E[1(N \geq i)E[1(C_i > w/\gamma_n)|N]]
$$

$$
\leq \frac{\gamma_n^{\alpha+\epsilon}}{w^{\alpha+\epsilon}} \sum_{i=1}^\infty E\left[C_i^{\alpha+\epsilon}1(N \geq i) \right] = E\left[\sum_{i=1}^N C_i^{\alpha+\epsilon} \right] \frac{\gamma_n^{\alpha+\epsilon}}{x^{\alpha+\epsilon/2}}. \qquad (22)
$$

Recall from the proof of Lemma 5.2 that $\gamma_n \leq K_{1,2}(\rho \vee \rho_{1+\delta})^{n/(1+\delta)}$ for some constant $K_{1,2} = K_{1,2}(\delta)$. It follows that (22) is bounded by

$$
\rho_{\alpha+\epsilon}\left(K_{1,2}(\rho \vee \rho_{1+\delta})^{n/(1+\delta)} \right)^{\alpha+\epsilon} \frac{1}{x^{\alpha+\epsilon/2}} \leq \rho_{\alpha+\epsilon}K_{1,2}^{\alpha+\epsilon}(\rho \vee \rho_{1+\delta})^n \frac{1}{x^{\alpha+\epsilon/2}},
$$

for any $x > 0$. Now, to bound the second probability in the statement, note that the same arguments used above give

$$
P(I_N(w/\gamma_n) = 0, I_N(x/T) \geq 1) \leq P(I_N(x/T) \geq 1) \leq \rho_{\alpha+\epsilon}T^{\alpha+\epsilon}\frac{1}{x^{\alpha+\epsilon}}.
$$

The convexity of $f(\theta) = \rho_\theta$ gives $\rho \vee \rho_{1+\delta} \leq \rho \vee \rho_\alpha$, from where it follows that there exists a constant $K_{3,1} = K_{3,1}(\delta, \epsilon, T)$ such that

$$P\left(I_N(w/\gamma_n) \geq 1\right) + P\left(I_N(w/\gamma_n) = 0, \, I_N(x/T) \geq 1\right)$$

$$\leq K_{3,1}\left((\rho \vee \rho_\alpha)^n + \frac{1}{x^{\epsilon/2}}\right) x^{-\alpha-\epsilon/2}$$

$$\leq K_{3,1}\left((\rho \vee \rho_\alpha)^n + \frac{1}{x^{\epsilon/2}}\right) \sup_{t \geq 1} \frac{1}{t^{\epsilon/2}L(t)} \cdot x^{-\alpha}L(x)$$

for all $x \geq 1$. $\quad \square$

Lemma 5.5. *Under the assumptions of Lemma 5.1, there exists a finite constant $K_4 = K_4(\delta)$ such that for all $x \geq 1$,*

$$P\left(Z_N > x/a_n\right) \leq K_4(\rho \vee \rho_\alpha)^n x^{-\alpha}L(x).$$

Proof. We use Potter's Theorem (see Theorem 1.5.6(iii) in [6]) to obtain that there exists a constant $x_0 = x_0(2, \delta) > 0$ such that for all $\min\{x, x/a_0\} \geq x_0$

$$P(Z_N > x/a_n) \leq \frac{(x/a_n)^{-\alpha}L(x/a_n)}{x^{-\alpha}L(x)} \cdot x^{-\alpha}L(x)$$

$$\leq 2\max\left\{\left(\frac{x/a_n}{x}\right)^{-\alpha+\delta}, \left(\frac{x/a_n}{x}\right)^{-\alpha-\delta}\right\} x^{-\alpha}L(x)$$

$$= 2\max\left\{\left(\frac{E[Q^+](\rho \vee \rho_{1+\delta})^{n/(1+\delta)}}{\delta^2}\right)^{\alpha-\delta},\right.$$

$$\left.\left(\frac{E[Q^+](\rho \vee \rho_{1+\delta})^{n/(1+\delta)}}{\delta^2}\right)^{\alpha+\delta}\right\} x^{-\alpha}L(x)$$

$$\leq \frac{2(E[Q^+] \vee 1)^{\alpha+\delta}}{\delta^{2(\alpha+\delta)}}(\rho \vee \rho_{1+\delta})^{(\alpha-\delta)n/(1+\delta)}x^{-\alpha}L(x).$$

The convexity of $f(\theta) = \rho_\theta$ and our choice of δ gives $(\rho \vee \rho_{1+\delta})^{(\alpha-\delta)n/(1+\delta)} \leq (\rho \vee \rho_\alpha)^n$, from where it follows that

$$P(Z_N > x/a_n) \leq \frac{2(E[Q^+] \vee 1)^{\alpha+\delta}}{\delta^{2(\alpha+\delta)}}(\rho \vee \rho_\alpha)^n x^{-\alpha}L(x) \triangleq K_{4,1}(\rho \vee \rho_\alpha)^n x^{-\alpha}L(x)$$

for all $x \geq \max\{x_0, a_0x_0\}$. For the values $1 \leq x \leq \max\{x_0, a_0x_0\}$ use Markov's inequality to obtain

$$P(Z_N > x/a_n) \leq \frac{a_n^{\alpha-\delta}}{x^{\alpha-\delta}} \leq K_{4,1}(\rho \vee \rho_\alpha)^n x^{-\alpha+\delta}$$

$$\leq K_{4,1} \sup_{1 \leq t \leq \max\{x_0, a_0x_0\}} \frac{t^\delta}{L(t)} \cdot x^{-\alpha}L(x).$$

Setting $K_4 = K_{4,1}\max\left\{1, \sup_{1 \leq t \leq \max\{x_0, a_0x_0\}} t^\delta L(t)^{-1}\right\}$ gives the statement of the lemma. $\quad \square$

We are now ready to give the proof of Lemma 5.1.

Proof of Lemma 5.1. First recall that $W_{n+1} \overset{\mathcal{D}}{=} \sum_{i=1}^{N} C_i W_{n,i}$, where the $W_{n,i}$ are i.i.d. having the same distribution as W_n and are independent of the vector $(N, C_1, C_2, \ldots)$. The idea of the proof

is to split $\{\sum_{i=1}^{N} C_i W_{n,i}^+ > x\}$ into several different events, and bound each of them separately. We proceed as follows,

$$P\left(W_{n+1}^+ > x\right) \leq P\left(\sum_{i=1}^{N} C_i W_{n,i}^+ > x\right)$$

$$\leq P\left(\sum_{i=1}^{N} C_i W_{n,i}^+ > x, Z_N \leq x/a_n\right) + P\left(Z_N > x/a_n\right)$$

$$\leq P\left(\sum_{i=1}^{N} C_i W_{n,i}^+ > x, Z_N \leq x/a_n, I_N(w/\gamma_n) = 0\right)$$
$$+ P\left(I_N(w/\gamma_n) \geq 1\right) + P\left(Z_N > x/a_n\right)$$

$$\leq P\left(\sum_{i=1}^{N} C_i W_{n,i}^+ > x, J_N(y) = 0, Z_N \leq x/a_n, I_N(w/\gamma_n) = 0\right) \quad (23)$$

$$+ P\left(\sum_{i=1}^{N} C_i W_{n,i}^+ > x, J_N(y) = 1, Z_N \leq x/a_n, I_N(w/\gamma_n) = 0\right) \quad (24)$$
$$+ P\left(J_N(y) \geq 2, Z_N \leq x/a_n, I_N(w/\gamma_n) = 0\right)$$
$$+ P\left(I_N(w/\gamma_n) \geq 1\right) + P\left(Z_N > x/a_n\right).$$

Note that the probability in (24) is bounded by

$$P\left(\sum_{i=1}^{N} C_i W_{n,i}^+ > x, J_N(y) = 1, J_N((1-\delta)x) = 0, Z_N \leq x/a_n, I_N(w/\gamma_n) = 0\right)$$
$$+ P\left(J_N((1-\delta)x) \geq 1, I_N(w/\gamma_n) = 0\right)$$
$$\leq P\left(\sum_{i=1}^{N} C_i W_{n,i}^+ 1(C_i W_{n,i}^+ \leq y) > \delta x, Z_N \leq x/a_n, I_N(w/\gamma_n) = 0\right)$$
$$+ P\left(J_N((1-\delta)x) \geq 1, I_N(x/T) = 0\right) + P\left(I_N(w/\gamma_n) = 0, I_N(x/T) \geq 1\right),$$

while (23) is bounded by

$$P\left(\sum_{i=1}^{N} C_i W_{n,i}^+ 1(C_i W_{n,i}^+ \leq y) > x, Z_N \leq x/a_n, I_N(w/\gamma_n) = 0\right).$$

It follows that

$$P(W_{n+1}^+ > x) \leq 2P\left(\sum_{i=1}^{N} C_i W_{n,i}^+ 1(C_i W_{n,i}^+ \leq y) > \delta x, Z_N \leq x/a_n, I_N(w/\gamma_n) = 0\right) \quad (25)$$

$$+ P\left(J_N(y) \geq 2, Z_N \leq x/a_n, I_N(w/\gamma_n) = 0\right) \quad (26)$$
$$+ P\left(I_N(w/\gamma_n) \geq 1\right) + P\left(I_N(w/\gamma_n) = 0, I_N(x/T) \geq 1\right) \quad (27)$$
$$+ P\left(Z_N > x/a_n\right) \quad (28)$$
$$+ P\left(J_N((1-\delta)x) \geq 1, I_N(x/T) = 0\right).$$

By Lemmas 5.2–5.5, the sum of the terms on the right-hand sides of (25)–(28) is bounded by

$$(2K_1 + K_2 + K_3 + K_4)(\rho \vee \rho_\alpha)^n x^{-\alpha} L(x) + K_3 x^{-\alpha - \epsilon/2} L(x)$$

for all $x \geq 1$ and all $0 \leq n \leq \frac{\epsilon}{2|\log(\rho \vee \rho_\alpha)|} \log x$, where K_1, K_2, K_3, K_4 are finite constants that only depend on ϵ, δ and T. Moreover, for this range of values of n we have

$$x^{-\epsilon/2} = (\rho \vee \rho_\alpha)^{\frac{\epsilon}{2|\log(\rho \vee \rho_\alpha)|} \log x} \leq (\rho \vee \rho_\alpha)^n.$$

Define $K_0 = K_0(\delta, \epsilon) = 2K_1 + K_2 + 2K_3 + K_4$ to obtain that

$$P(W_{n+1}^+ > x) \leq K_0(\rho \vee \rho_\alpha)^n x^{-\alpha} L(x) + P\left(J_N((1 - \delta)x) \geq 1, I_N(x/T) = 0\right). \quad (29)$$

To bound (29) use the union bound to obtain

$$P\left(J_N((1 - \delta)x) \geq 1, I_N(x/T) = 0\right)$$
$$= E\left[1(I_N(x/T) = 0) \cdot 1\left(\bigcup_{i=1}^N \{C_i W_{n,i}^+ > (1 - \delta)x\}\right)\right]$$
$$\leq E\left[1(I_N(x/T) = 0) \sum_{i=1}^N 1\left(C_i W_{n,i}^+ > (1 - \delta)x\right)\right], \quad (30)$$

which completes the proof. $\quad\square$

We can now give the proof of Proposition 3.2, the main technical contribution of the paper.

Proof of Proposition 3.2. Recall that $\overline{G}(x) = P(Z_N > x)$. Note that it is enough to prove the proposition for all $x \geq x_1$ for some $x_1 = x_1(\eta, \epsilon) \geq 1$, since for all $1 \leq x \leq x_1$ and $n \geq 1$,

$$P(W_n^+ > x) = \frac{P(W_n^+ > x)}{\eta^n \overline{G}(x)} \eta^n \overline{G}(x)$$
$$\leq \frac{E[Q^+] \rho^n x^{-1}}{\eta^n \overline{G}(x)} \eta^n \overline{G}(x) \quad \text{(by Markov's inequality)}$$
$$\leq \sup_{1 \leq t \leq x_1} \frac{E[Q^+]}{t \overline{G}(t)} \cdot \eta^n \overline{G}(x).$$

Next, choose $0 < \delta < \min\{(\alpha - 1)/2, \epsilon, 1/2\}$ such that

$$\rho_\alpha \left(\delta + (1 - \delta)^{-\alpha - 1}\right) \leq \eta. \quad (31)$$

Now note that by Potter's Theorem (see Theorem 1.5.6(iii) in [6]), there exists a constant $x_0 = x_0(2, \delta) > 0$ such that

$$E\left[\sum_{i=1}^N \frac{\overline{G}((1 - \delta)x/C_i)}{\overline{G}(x)}\right]$$
$$\leq E\left[\sum_{i=1}^N 2(1 - \delta)^{-\alpha} C_i^\alpha \max\{((1 - \delta)/C_i)^{-\delta}, ((1 - \delta)/C_i)^\delta\}\right]$$
$$\leq 2(1 - \delta)^{-\alpha - \delta}(\rho_{\alpha - \delta} + \rho_{\alpha + \delta}) < \infty$$

for all $x \geq x_0$. And for $1 \leq x \leq x_0$ Markov's inequality gives

$$E\left[\sum_{i=1}^{N} \frac{\overline{G}((1-\delta)x/C_i)}{\overline{G}(x)}\right] \leq \frac{1}{\overline{G}(x)} E\left[\sum_{i=1}^{N} \frac{E[Z_N^{\alpha-\delta}]C_i^{\alpha-\delta}}{(1-\delta)^{\alpha-\delta}x^{\alpha-\delta}}\right]$$

$$\leq \frac{E[Z_N^{\alpha-\delta}]\rho_{\alpha-\delta}}{(1-\delta)^{\alpha-\delta}} \sup_{1 \leq t \leq x_0} \frac{t^{-\alpha+\delta}}{\overline{G}(t)} < \infty.$$

Hence, by dominated convergence,

$$\lim_{x\to\infty} E\left[\sum_{i=1}^{N} \frac{\overline{G}((1-\delta)x/C_i)}{\overline{G}(x)}\right] = E\left[\sum_{i=1}^{N} \lim_{x\to\infty} \frac{\overline{G}((1-\delta)x/C_i)}{\overline{G}(x)}\right] = (1-\delta)^{-\alpha}\rho_\alpha.$$

It follows that there exists $x_1 = x_1(\delta) \geq 1$ for which

$$E\left[\sum_{i=1}^{N} \frac{\overline{G}((1-\delta)x/C_i)}{\overline{G}(x)}\right] \leq (1-\delta)^{-\alpha-1}\rho_\alpha \tag{32}$$

for all $x \geq x_1$. Set $T = 2x_1$.

Now, by Lemma 5.1, there exists a finite constant $K_0 > 0$ (that does not depend on n) such that

$$P(W_{n+1}^+ > x) \leq K_0(\rho \vee \rho_\alpha)^n \overline{G}(x)$$

$$+ E\left[1(I_N(x/T) = 0) \sum_{i=1}^{N} 1\left(C_i W_{n,i}^+ > (1-\delta)x\right)\right]$$

for all $x \geq 1$ and $0 \leq n \leq \frac{\epsilon}{2|\log(\rho\vee\rho_\alpha)|} \log x$. Let $K_1 = (\delta\rho_\alpha)^{-1}K_0$ to obtain

$$P(W_{n+1}^+ > x) \leq K_1 \delta\rho_\alpha \eta^n \overline{G}(x) + E\left[1(I_N(x/T) = 0) \sum_{i=1}^{N} 1\left(C_i W_{n,i}^+ > (1-\delta)x\right)\right] \tag{33}$$

for all $x \geq 1$ and $0 \leq n \leq \frac{\epsilon}{2|\log(\rho\vee\rho_\alpha)|} \log x$.

Now we go on to derive bounds for $P(W_n^+ > x)$ for different ranges of n. For the values $1 \leq n \leq \frac{\epsilon}{2|\log(\rho\vee\rho_\alpha)|} \log x$ we proceed by induction. Let $\mathcal{F} = \sigma(N, C_1, C_2, \ldots)$. Define

$$K_2 = \max\left\{K_1, K_1\delta + E[(Q^+)^{\alpha+\epsilon}]\frac{\rho_{\alpha+\epsilon}}{\eta} \sup_{t\geq 1} \frac{1}{t^\epsilon L(t)}\right\}.$$

For $n = 1$, we have by (33),

$$P(W_1^+ > x) \leq K_1 \delta\rho_\alpha \overline{G}(x) + E\left[1(I_N(x/T) = 0) \sum_{i=1}^{N} 1\left(C_i W_{0,i}^+ > (1-\delta)x\right)\right],$$

where $W_{0,i}^+ = Q_i^+$ and $\{Q_i^+\}$ are independent of $(N, C_1, C_2, \ldots)$. By conditioning on $\mathcal{F}$ we get

$$E\left[1(I_N(x/T) = 0) \sum_{i=1}^{N} 1\left(C_i W_{0,i}^+ > (1-\delta)x\right)\right]$$

$$= E\left[1(I_N(x/T) = 0) \sum_{i=1}^{N} E[1\left(C_i Q_i^+ > (1-\delta)x\right)|\mathcal{F}]\right]$$

$$\leq E\left[\sum_{i=1}^{N} \frac{E[(C_i Q_i^+)^{\alpha+\epsilon}|C_i]}{x^{\alpha+\epsilon}}\right] \quad \text{(by Markov's inequality)}$$

$$= E[(Q^+)^{\alpha+\epsilon}]\rho_{\alpha+\epsilon}x^{-\alpha-\epsilon}$$

$$\leq E[(Q^+)^{\alpha+\epsilon}]\rho_{\alpha+\epsilon}\sup_{t\geq 1}\frac{1}{t^\epsilon L(t)}\overline{G}(x).$$

It follows that

$$P(W_1^+ > x) \leq \left(K_1\delta + E[(Q^+)^{\alpha+\epsilon}]\frac{\rho_{\alpha+\epsilon}}{\eta}\sup_{t\geq 1}\frac{1}{t^\epsilon L(t)}\right)\eta\overline{G}(x) \leq K_2\eta\overline{G}(x)$$

for all $x \geq 1$. Suppose now that

$$\overline{G}_n(x) \triangleq P(W_n^+ > x) \leq K_2\eta^n\overline{G}(x) \tag{34}$$

for all $x \geq x_1$.

Let $2 \leq n \leq \frac{\epsilon}{2|\log(\rho\vee\rho_\alpha)|}\log x$. Then, by the induction hypothesis (34), we have for all $x \geq x_1$,

$$E\left[1(I_N(x/T) = 0)\sum_{i=1}^{N}1\left(C_i W_{n,i}^+ > (1-\delta)x\right)\right]$$

$$= E\left[1(I_N(x/T) = 0)\sum_{i=1}^{N}E\left[1\left(C_i W_{n,i}^+ > (1-\delta)x\right)\Big|\mathcal{F}\right]\right]$$

$$= E\left[1(I_N(x/T) = 0)\sum_{i=1}^{N}\overline{G}_n((1-\delta)x/C_i)\right]$$

$$\leq K_2\eta^n E\left[1(I_N(x/T) = 0)\sum_{i=1}^{N}\overline{G}((1-\delta)x/C_i)\right]$$

$$= K_2\eta^n\overline{G}(x)E\left[1(I_N(x/T) = 0)\sum_{i=1}^{N}\frac{\overline{G}((1-\delta)x/C_i)}{\overline{G}(x)}\right]$$

$$\leq K_2\eta^n(1-\delta)^{-\alpha-1}\rho_\alpha\overline{G}(x),$$

where in the last inequality we used (32). Then, by (33)

$$P(W_{n+1}^+ > x) \leq K_1\delta\rho_\alpha\eta^n\overline{G}(x) + K_2\eta^n(1-\delta)^{-\alpha-1}\rho_\alpha\overline{G}(x)$$

$$\leq K_2\left(\delta + (1-\delta)^{-\alpha-1}\right)\rho_\alpha\eta^n\overline{G}(x)$$

$$\leq K_2\eta^{n+1}\overline{G}(x),$$

for all $x \geq x_1$.

Finally, for $n \geq \frac{\epsilon}{2|\log(\rho\vee\rho_\alpha)|}\log x$, we use the moment estimates for W_n. Define

$$\varepsilon = \frac{\eta}{\rho\vee\rho_\alpha} - 1 > 0 \quad \text{and} \quad \kappa = \frac{\epsilon\log(1+\varepsilon)}{2|\log(\rho\vee\rho_\alpha)|}.$$

Choose $0 < s < \min\{\kappa/2, \alpha - 1\}$. Then, by Markov's inequality and Proposition 2.2, we have

$$
\begin{aligned}
P(W_n^+ > x) &\le E[(W_n^+)^{\alpha-s}]x^{-\alpha+s} \\
&\le K_{\alpha-s}(\rho \vee \rho_{\alpha-s})^n x^{-\alpha+s} \\
&\le K_{\alpha-s}(1+\varepsilon)^{-n}\eta^n x^{-\alpha+s} \\
&\le K_{\alpha-s}x^{-\log(1+\varepsilon)\frac{\epsilon}{2|\log(\rho\vee\rho_\alpha)|}}\eta^n x^{-\alpha+s} \\
&= K_{\alpha-s}\eta^n x^{-\alpha-\kappa+s}
\end{aligned}
\tag{35}
$$

for all $x > 0$. Our choice of s now gives

$$
P(W_n^+ > x) \le K_{\alpha-s}\eta^n x^{-\alpha-\kappa/2} \le K_{\alpha-s}\sup_{t\ge1}\frac{t^{-\kappa/2}}{L(t)}\cdot\eta^n\overline{G}(x) \triangleq K_3\eta^n\overline{G}(x)
$$

for all $x \ge 1$.

We have thus shown that

$$
P(W_n^+ > x) \le \max\{K_2, K_3\}\eta^n\overline{G}(x)
$$

for all $x \ge x_1$ and $n \ge 1$. $\square$

We end this section with a sketch of the proof of Proposition 4.2. As mentioned before, the proofs of the other results presented in Section 4 have been omitted since they are very similar to those from Section 3.

Sketch of the proof of Proposition 4.2. Follow the proof of Proposition 3.2 up to inequality (32) substituting $\overline{G}(x) = P(Z_N > x)$ with $\overline{F}(x) \triangleq P(Q > x) = x^{-\alpha}L(x)$. Now note that by Markov's inequality

$$
P(Z_N > x) \le E[Z_N^{\alpha+\epsilon}]x^{-\alpha-\epsilon}
$$

for all $x > 0$, so we can use Lemma 5.1 to obtain

$$
\begin{aligned}
P(W_{n+1}^+ > x) &\le K_0(\rho \vee \rho_\alpha)^n E[Z_N^{\alpha+\epsilon}]x^{-\alpha-\epsilon} \\
&\quad + E\left[1(I_N(x/T) = 0)\sum_{i=1}^N 1(C_i W_{n,i}^+ > (1-\delta)x)\right]
\end{aligned}
$$

for all $x \ge 1$ and all $0 \le n \le \frac{\epsilon}{2|\log(\rho\vee\rho_\alpha)|}\log x$; $K_0 > 0$ is a constant that does not depend on n. Let $K_1 = (\delta\rho_\alpha)^{-1}K_0 E[Z_N^{\alpha+\epsilon}]\sup_{t\ge1}t^{-\epsilon}/L(t)$ to derive

$$
P(W_{n+1}^+ > x) \le K_1\delta\rho_\alpha\eta^n\overline{F}(x) + E\left[1(I_N(x/T) = 0)\sum_{i=1}^N 1(C_i W_{n,i}^+ > (1-\delta)x)\right] \tag{36}
$$

for all $x \ge 1$ and all $0 \le n \le \frac{\epsilon}{2|\log(\rho\vee\rho_\alpha)|}\log x$.

Now define $\mathcal{F} = \sigma(N, C_1, C_2, \ldots)$ and $K_2 = \max\{K_1, 1\}$. For the values $0 \le n \le \frac{\epsilon}{2|\log(\rho\vee\rho_\alpha)|}\log x$ we proceed by induction. For $n = 1$ we have $W_{0,i}^+ = Q_i^+$, with the $\{Q_i^+\}$ independent of $(N, C_1, C_2, \ldots)$. By conditioning on $\mathcal{F}$ and using (32) (with $\overline{G}(x)$ substituted by $\overline{F}(x)$), we get

$$E\left[1(I_N(x/T)=0)\sum_{i=1}^{N}1(C_i W_{0,i}^+ > (1-\delta)x)\right]$$

$$= E\left[1(I_N(x/T)=0)\sum_{i=1}^{N}\overline{F}((1-\delta)x/C_i)\right]$$

$$\le (1-\delta)^{-\alpha-1}\rho_\alpha \overline{F}(x).$$

It follows that

$$P(W_1^+ > x) \le K_1\delta\rho_\alpha \overline{F}(x) + (1-\delta)^{-\alpha-1}\rho_\alpha \overline{F}(x) \le K_2\eta\overline{F}(x)$$

for all $x \ge 1$.

The rest of the proof continues exactly as that of Proposition 3.2 with $\overline{G}(x)$ substituted by $\overline{F}(x)$. $\square$

Acknowledgments

Appendix. Some results for weighted random sums

We include in this appendix two results related to the asymptotic behavior of randomly weighted and randomly stopped sums. The first one is a quick corollary of a theorem from [22] that allows the addition of the Q term for the case where Z_N has a regularly varying distribution. The second one also uses some of the results from [22], but is more involved since it refers to the case where Q has a regularly varying distribution. Both of these results may be of independent interest.

Theorem A.1. *Let $\{X_i\}$ be a sequence of i.i.d. random variables with common distribution $\overline{F} \in \mathcal{R}_{-\alpha}, \alpha > 1, E[(X_1^-)^{1+\epsilon}] < \infty$ for some $0 < \epsilon < \alpha - 1$, and $E[X_1] > 0$. Assume further that $(Q, N, C_1, C_2, \ldots)$ is a random vector, independent of the $\{X_i\}$, with $N \in \mathbb{N} \cup \{\infty\}, C_i \ge 0$ for all $i \in \mathbb{N}$, and $Q \in \mathbb{R}$. Then, if $Z_N = \sum_{i=1}^{N}C_i$ satisfies $P(Z_N > x) \sim cP(X_1 > x)$ for some $c > 0$, $E\left[\sum_{i=1}^{N}C_i^{\alpha+\epsilon}\right] < \infty$ and $E[|Q|^{\alpha+\epsilon}] < \infty$, we have*

$$P\left(\sum_{i=1}^{N}C_i X_i + Q > x\right) \sim \left(E\left[\sum_{i=1}^{N}C_i^\alpha\right] + c(E[X_1])^\alpha\right)\overline{F}(x)$$

as $x \to \infty$.

Proof. Let $S_N = \sum_{i=1}^{N}C_i X_i$, and note that since $\alpha - \epsilon > 1$, the inequality $\sum_{i=1}^{k}y_i^\beta \le \left(\sum_{i=1}^{k}y_i\right)^\beta$ for $y_i \ge 0$ and any $\beta \ge 1$ gives $E\left[\sum_{i=1}^{N}C_i^{\alpha-\epsilon}\right] \le E\left[Z_N^{\alpha-\epsilon}\right]$, which is finite by the assumption $P(Z_N > x) \sim c\overline{F}(x)$. Then, by Theorem 2.5 in [22] and the remark after it,

$$P(S_N + Q > x) \le P(S_N + Q > x, Q \le x/\log x) + P(Q > x/\log x)$$

$$\le P(S_N > x - x/\log x) + \frac{E[|Q|^{\alpha+\epsilon}]}{(x/\log x)^{\alpha+\epsilon}} \quad \text{(by Markov's inequality)}$$

$$\sim E\left[\sum_{i=1}^{N} C_i^{\alpha}\right] \overline{F}(x - x/\log x)$$

$$+ P(Z_N > (x - x/\log x)/E[X_1]) + o\left(\overline{F}(x)\right)$$

$$\sim E\left[\sum_{i=1}^{N} C_i^{\alpha}\right] \overline{F}(x) + c(E[X_1])^{\alpha} \overline{F}(x).$$

For the lower bound, the same arguments give

$$
\begin{aligned}
P(S_N + Q > x) &\geq P(S_N + Q > x, Q \geq -x/\log x) \\
&\geq P(S_N > x + x/\log x) - P(Q < -x/\log x) \\
&\geq P(S_N > x + x/\log x) - \frac{E[|Q|^{\alpha+\epsilon}]}{(x/\log x)^{\alpha+\epsilon}} \\
&\sim E\left[\sum_{i=1}^{N} C_i^{\alpha}\right] \overline{F}(x) + c(E[X_1])^{\alpha} \overline{F}(x). \quad \square
\end{aligned}
$$

Theorem A.2. *Let $\{X_i\}$ be a sequence of i.i.d. random variables with common distribution $\overline{F} \in \mathcal{R}_{-\alpha}, \alpha > 1, E[(X_1^-)^{1+\epsilon}] < \infty$ for some $\epsilon > 0$. Assume further that $(Q, N, C_1, C_2, \ldots)$ is a random vector, independent of the $\{X_i\}$, with $N \in \mathbb{N} \cup \{\infty\}, C_i \geq 0$ for all $i \in \mathbb{N}$, and $Q \in \mathbb{R}$. Then, if $P(Q > x) \sim cP(X_1 > x)$ for some $c > 0$, and $Z_N = \sum_{i=1}^{N} C_i$ satisfies $E\left[Z_N^{\alpha+\epsilon}\right] < \infty$, we have*

$$P\left(\sum_{i=1}^{N} C_i X_i + Q > x\right) \sim \left(E\left[\sum_{i=1}^{N} C_i^{\alpha}\right] + c\right) \overline{F}(x)$$

as $x \to \infty$.

Proof. Let $S_N = \sum_{i=1}^{N} C_i X_i$ and define $J_N(t) = \#\{1 \leq i < N+1 : C_i X_i > t\}$. Assume that $0 < \epsilon < \alpha - 1$ and set $\nu = \epsilon/(2(\alpha + \epsilon)), \gamma = \left(E[|X_1|^{1+\epsilon}]\right)^{1/(1+\epsilon)}, w = x^{1-\nu}/\gamma, y = x/\log x$ and $\delta = 1/\sqrt{\log x}$. Also note that

$$E\left[\sum_{i=1}^{N} C_i^{\alpha-\epsilon}\right] \leq E\left[Z_N^{\alpha-\epsilon}\right] \leq \left(E\left[Z_N^{\alpha+\epsilon}\right]\right)^{\frac{\alpha-\epsilon}{\alpha+\epsilon}} < \infty.$$

Then,

$$
\begin{aligned}
P(S_N + Q > x) &\leq P(S_N + Q > x, S_N > (1-\delta)x) \\
&\quad + P(S_N + Q > x, S_N \leq (1-\delta)x, Q > (1-\delta)x) \\
&\quad + P(S_N + Q > x, S_N \leq (1-\delta)x, Q \leq (1-\delta)x) \\
&\leq P(S_N > (1-\delta)x) + P(Q > (1-\delta)x) \qquad\qquad\qquad\text{(A.1)} \\
&\quad + P(S_N + Q > x, S_N \leq (1-\delta)x, \delta x < Q \leq (1-\delta)x). \quad\text{(A.2)}
\end{aligned}
$$

By Theorem 2.3 and the remark following Theorem 2.5 in [22], we have that (A.1) is equal to

$$E\left[\sum_{i=1}^{N} C_i^{\alpha}\right] \overline{F}(x) + c\overline{F}(x) + o\left(\overline{F}(x)\right)$$

as $x \to \infty$. To analyze (A.2) first note that it is bounded by

$$
\begin{aligned}
P\left(S_N > \delta x, Q > \delta x\right) &\le P\left(S_N > \delta x, Q > \delta x, Z_N \le w\right) + P(Z_N > w) \\
&\le P\left(S_N > \delta x, Q > \delta x, Z_N \le w, J_N(y) = 0\right) \\
&\quad + P\left(Q > \delta x, Z_N \le w, J_N(y) \ge 1\right) + P(Z_N > w) \\
&\le P\left(\sum_{i=1}^{N} C_i X_i^+ > \delta x, J_N(y) = 0, Z_N \le w\right) \\
&\quad + P\left(Q > \delta x, Z_N \le w, J_N(y) \ge 1\right) + P(Z_N > w) \\
&\le P\left(\sum_{i=1}^{N} C_i X_i^+ 1(C_i X_i^+ \le y) > \delta x, Z_N \le w\right) \quad\quad (A.3) \\
&\quad + P\left(Q > \delta x, Z_N \le w, J_N(y) \ge 1\right) \quad\quad\quad\quad\quad (A.4) \\
&\quad + P(Z_N > w).
\end{aligned}
$$

Now, by Lemma 3.4 in [22] (note that $Z_N \le w$ implies $I_N(w) = \#\{1 \le i < N+1 : C_i > w\} = 0$), (A.3) is bounded by $K x^{-h}$ for any $h > 0$, in particular, for $h = \alpha + \epsilon$, from where it follows that it is $o\left(\overline{F}(x)\right)$. Here and in the remainder of the proof $K > 0$ is a generic constant, not necessarily the same from one line to the next. To analyze (A.4) let $\mathcal{F} = \sigma(Q, N, C_1, C_2, \ldots)$ and note that we can write the probability as

$$
\begin{aligned}
&E\left[1(Q > \delta x, Z_N \le w) E\left[1(J_N(y) \ge 1)|\mathcal{F}\right]\right] \\
&\le E\left[1(Q > \delta x, Z_N \le w) \sum_{i=1}^{N} E\left[1(C_i X_i > y)|\mathcal{F}\right]\right] \quad \text{(by the union bound)} \\
&\le \frac{E[|X_1|^{1+\epsilon}]}{y^{1+\epsilon}} E\left[1(Q > \delta x, Z_N \le w) \sum_{i=1}^{N} C_i^{1+\epsilon}\right] \quad \text{(by Markov's inequality)} \\
&\le \frac{K}{y^{1+\epsilon}} E\left[1(Q > \delta x, Z_N \le w) Z_N^{1+\epsilon}\right] \le \frac{K w^{1+\epsilon}}{y^{1+\epsilon}} P(Q > \delta x) \\
&\le \frac{K (\log x)^{1+\epsilon}}{x^{(1+\epsilon)\epsilon\nu}} \overline{F}(\delta x) \le \frac{K (\log x)^{1+\epsilon}}{x^{(1+\epsilon)\epsilon\nu} \delta^{\alpha+\epsilon}} \overline{F}(x) \\
&= \frac{K (\log x)^{1+\alpha/2+3\epsilon/2}}{x^{(1+\epsilon)\epsilon\nu}} \overline{F}(x) = o\left(\overline{F}(x)\right),
\end{aligned}
$$

where in the sixth inequality we used Potter's Theorem (see Theorem 1.5.6 in [6]). Finally, from Markov's inequality we get

$$
P(Z_N > w) \le \frac{E[Z_N^{\alpha+\epsilon}]}{w^{\alpha+\epsilon}} \le \frac{K}{x^{(1-\nu)(\alpha+\epsilon)}} = \frac{K}{x^{\alpha+\epsilon/2}} = o\left(\overline{F}(x)\right).
$$

We have thus shown that (A.2) is $o\left(\overline{F}(x)\right)$, and the upper bound follows.

For the lower bound we have that

$$
\begin{aligned}
P\left(S_N + Q > x\right) &\ge P\left(S_N + Q > x, Z_N \le w, S_N > (1+\delta)x\right) \\
&\quad + P\left(S_N + Q > x, Z_N \le w, S_N \le (1+\delta)x, Q > (1+\delta)x\right) \\
&= P\left(S_N > (1+\delta)x, Z_N \le w\right) + P\left(Q > (1+\delta)x, Z_N \le w\right) \quad (A.5) \\
&\quad - P\left(S_N + Q \le x, Z_N \le w, S_N > (1+\delta)x\right) \quad\quad\quad\quad\quad (A.6) \\
&\quad - P\left(S_N + Q \le x, Z_N \le w, S_N \le (1+\delta)x, Q > (1+\delta)x\right) \quad (A.7) \\
&\quad - P\left(Z_N \le w, S_N > (1+\delta)x, Q > (1+\delta)x\right). \quad\quad\quad\quad (A.8)
\end{aligned}
$$

Note that (A.5) is bounded from below by

$$P(S_N > (1+\delta)x) + P(Q > (1+\delta)x) - 2P(Z_N > w) = E\left[\sum_{i=1}^{N} C_i^{\alpha}\right]\overline{F}(x)$$
$$+ c\overline{F}(x) + o\left(\overline{F}(x)\right),$$

by the same arguments used for the upper bound. Also note that we can bound the sum of the probabilities in (A.7) and (A.8) by

$$P(S_N \leq -\delta x, Z_N \leq w, Q > x) + P(S_N > \delta x, Z_N \leq w, Q > x)$$
$$\leq 2P(Z_N \leq w, |S_N| \geq \delta x, Q > x)$$
$$= 2E\left[1(Z_N \leq w, Q > x)E\left[1(|S_N| \geq \delta x)|\mathcal{F}\right]\right]$$
$$\leq \frac{2}{\delta x}E\left[1(Z_N \leq w, Q > x)E\left[|S_N||\mathcal{F}\right]\right] \quad \text{(by Markov's inequality)}$$
$$\leq \frac{2E[|X_1|]}{\delta x}E\left[1(Z_N \leq w, Q > x)Z_N\right]$$
$$\leq \frac{Kw}{\delta x}P(Q > x) \leq \frac{K(\log x)^{1/2}}{x^{\nu}}\overline{F}(x) = o\left(\overline{F}(x)\right).$$

It only remains to analyze (A.6). Let $\kappa = \nu^2$ and note that the probability in (A.6) is bounded by

$$P(S_N + Q \leq x, Z_N \leq w, S_N > (1+\delta)x, J_N(\kappa x) = 0)$$
$$+ P(S_N + Q \leq x, Z_N \leq w, S_N > (1+\delta)x, J_N(\kappa x) \geq 1)$$
$$\leq P\left(\sum_{i=1}^{N} C_i X_i 1(C_i X_i \leq \kappa x) > (1+\delta)x, Z_N \leq w\right) \tag{A.9}$$
$$+ P(Q < -\delta x, Z_N \leq w, S_N > (1+\delta)x, J_N(\kappa x) \geq 1). \tag{A.10}$$

By Lemma 3.2 in [22], with $u = x^{1-\nu}$, $v = \kappa x$, $z = x$, $\eta = 1 + \epsilon$ and $A = (-\infty, w]$ (note that $Z_N \leq w$ implies $I_N(w) = 0$), (A.9) is bounded by

$$E\left[1(Z_N \leq w)e^{-\frac{\epsilon}{\kappa x}\log(\kappa x^{\nu})\left(x - \left(E[X_1] + \frac{K\gamma}{\log(\kappa x^{\nu})}\right)^{+}Z_N\right)}\right] \leq Ke^{-\frac{\epsilon}{\kappa}\log(\kappa x^{\nu})}$$
$$\leq \frac{K}{x^{2(\alpha+\epsilon)}} = o\left(\overline{F}(x)\right).$$

As for (A.10) use Potter's Theorem (see Theorem 1.5.6(iii) in [6]) to obtain,

$$P(Q < -\delta x, Z_N \leq w, J_N(\kappa x) \geq 1)$$
$$\leq E\left[1(Q < -\delta x, Z_N \leq w)\sum_{i=1}^{N}\overline{F}(\kappa x/C_i)\right] \quad \text{(by the union bound)}$$
$$\leq K\overline{F}(x)E\left[1(Q < -\delta x)\sum_{i=1}^{N} C_i^{\alpha+\epsilon}\right] \quad \text{(by Potter's Theorem)}$$
$$\leq K\overline{F}(x)E\left[1(Q < -\delta x)Z_N^{\alpha+\epsilon}\right]$$
$$= o\left(\overline{F}(x)\right),$$

where in the last step we used dominated convergence ($E[Z_N^{\alpha+\epsilon}] < \infty$) to see that $E\left[1(Q < -\delta x)Z_N^{\alpha+\epsilon}\right] \to 0$ as $x \to \infty$. $\quad\square$

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